MATH 562, SPRING 2021–FOURTH HOMEWORK SET. (Turn in 2 problems from 1-5 and 2 problems from 6-10.) Posted 4/10/21, due 4/19/21.

Note: all covering spaces and covered spaces assumed to be connected and locally path-connected.

1. Give examples of:
(a) A continuous bijection which is not a local homeomorphism;
(b) A continuous surjection $f : R \to R$ so that $f^{-1}(y)$ is discrete for all y, but f is not locally injective.

2. (a) Prove there does not exist $f : R \to R$ continuous, so that $f^{-1}(y)$ has exactly two points, for all $y \in R$.
(b) Let $p(z) = 2z^3 - 9z^2 + 12 + 1$. Find finite sets $F_1, F_2 \subset \mathbb{C}$, so that $p : \mathbb{C} \setminus F_1 \to \mathbb{C} \setminus F_2$ is a 3-sheeted covering.

3. Let $p : \tilde{X} \to X$ be a covering map, $a, b : I \to X$ two freely homotopic closed curves. If a has a closed lift $\tilde{a}$, then also b has a closed lift $\tilde{b}$, freely homotopic to $\tilde{a}$ in $\tilde{X}$.

4. Let U be the set of quaternions $w = t + xi + yj + zk$ such that $t > 0$, X the set of real quaternions ≤ 0, $V = R^4 \setminus X$. Show that $f : U \to V$, $f(w) = w^2$, is a local diffeomorphism, surjective and proper, hence a global diffeomorphism from U to V.

5. Let X be the figure eight, $\tilde{X}$ the subset of the upper half-plane made up of the horizontal axis, together with circles of radius 1/3 tangent to it at the points of integral abscissa. Define a covering map from $\tilde{X}$ to X. Is this covering regular? With x_0 the origin in R^2, describe the conjugacy class of $\pi_1(X, x_0)$ defined by this covering, and determine $G = Aut(\tilde{X}|X)$.

6. Let X be an arbitrary topological space. Given the cover $p : R \to S^1, p(t) = e^{it}$, prove that a continuous map $f : X \to S^1$ has a lift relative to p if, and only if, f is homotopic to a constant.

7. (a) Any continuous map from S^2 to T^2 is homotopic to a constant.
(b) If $n \geq 2$, any continuous map $f : P^n \to S^1$ is homotopic to a constant.

8. Let M be a compact orientable surface of genus $g > 1$. Prove there exists $f : M \to S^1$ continuous, not homotopic to a constant.

9. Let X be the figure eight, $\tilde{X}$ the grid in R^2 of points in the plane with at least one integer coordinate. Define a covering map $p : \tilde{X} \to X$. so that, fixing $\tilde{x_0} \in p^{-1}(x_0)$, the subgroup $H(\tilde{x_0})$ is the commutator subgroup of $\pi_1(X, x_0)$, and $Aut(\tilde{X}|X) = \mathbb{Z} \oplus \mathbb{Z}$.

10. Find the universal covers of: (i) the wedge of S^1 and S^2; (ii) the wedge of S^1 and T^2.

11. Let $(X, x_0), (Y, y_0)$ be spaces with basepoint, both locally basepoint-contractible (for example, manifolds). If both spaces are simply connected, then so is their wedge $(X \vee Y, z_0)$. (Example: two spheres of dimension at least two.)