MATH 562, SPRING 2021–FIFTH HOMEWORK SET.

**1.** (KA) [Milnor 1] Every complex polynomial of degree $n$ gives rise to a smooth map from $S^2$ to itself, of degree $n$.

**2.** (BA) [Milnor 1] If $m < p$, show that every smooth map $M^m \to S^p$ is homotopic to a constant.

**3.** (EB) [Milnor 1] Show that any (smooth) map $S^n \to S^n$ with degree different from $(-1)^{n+1}$ must have a fixed point.

The next problems [GP, p.144] deal with the *oriented winding number:* Given $X$ compact oriented with dimension $n$, and $f : X \to R^{n+1}$ , let $z \in R^{n+1} \setminus f(X)$. Define:

$$W(f,z) = deg(u), \qquad u(x) = \frac{f(x) - z}{|f(x) - z|}.$$

**4.** (PG) (i) [ no.1, in GP p.144]. Let $f : U \to R^k$ smooth, $U \subset R^k$ open, $x \in U$ a regular point, $\partial f : \partial B \to R^k$ the restriction of $f$ to the boundary of a small ball with center $x$.

Then $W(\partial f, z) = \pm 1$, depending on whether $f$ preserves (+1) or reverses (-1) orientation at $x$.

(ii) [2 in G-P] $f : B \to R^k$ smooth, $B \subset R^k$ closed ball, $z$ a regular value of $f$ with no preimages on the boundary sphere $\partial B$, $\partial f$ the restriction of $f$ to $\partial B$. Prove that the number of preimages of $z$ (counted with orientation) equals $W(\partial f, z)$. (See problem 2 on p.87, or the hint.)

**5.**(JH) [3 in G-P] $B \subset R^k$ closed ball, $f : R^k \setminus int(B) \to Y$ smooth map defined outside the open ball.

Show that if the restriction $\partial f$ of $f$ to the boundary of the ball is homotopic to a constant, then $f$ extends to a smooth map from $R^k$ to $Y$.

The next two problems deal with a special case of Hopf's theorem: a smooth map from $S^k$ to $S^k$ of degree zero is homotopic to a constant. This is proved by induction on $k$, the case $k = 1$ being already understood.

**6.**(TI) [5 in GP, p.145]. $f : R^k \to R^k$ smooth, with 0 as a regular value. Suppose $f^{-1}(0)$ is finite, with the number of preimage points adding to zero when counted with orientation. Assuming Hopf's theorem for maps of $(k-1)$-dimensional spheres, show there exists a smooth map $g : R^k \to R^k \setminus \{0\}$, coinciding with $f$ outside a compact set.

**7.** (JP) (i) [4 and 6 in GP, p. 145] Using the previous problem (and the hint in [G-P]) prove Hopf's theorem for maps $S^k \to S^k$ of degree 0.

(ii) As a corollary, prove that any smooth map $f : S^k \to R^{k+1} \setminus \{0\}$ with winding number 0 about the origin is homotopic to a constant.

The next two problems complete the proof of Hopf's theorem in the general case.

**8.** (MS) [7 in GP p. 146] Let $W$ be a compact manifold with boundary, $f : \partial W \to R^{k+1}$ any smooth map. Prove that $f$ may be extended (smoothly) to all of $W$.

**9.** (BW) [GP 8 and 9, p. 146] (i) Prove the Extension Theorem: If $W$ is a compact manifold with boundary of dimension $k+1$ and $f : \partial W \to S^k$ is a smooth map of degree 0, then $f$ extends smoothly to $F : W \to S^k$.

(ii) Prove *Hopf's theorem:* Two maps of a compact, oriented $k$-dimensional manifold $M$ (without boundary) to $S^k$ with the same degree are homotopic.

**10.** (SW) [G-P 10, p.146] Let $X$ be a vector field on $R^n$ with finitely many singularities so that the sum of the indices of its singularities is zero. Show that there exists a vector field $Y$ on $R^n$ with no singularities, equal to $X$ outside a compact set.

(This is the first step in the proof that compact oriented manifolds with zero Euler characteristic admit non-vanishing vector fields.)