1. Semi-continuous functions on a manifold. Let X be a topological space. A real function $f : X \to R$ is *lower semi-continuous* at $a \in X$ if for each $\epsilon > 0$ there exists a neighborhood V of a so that, for all $x \in V$, $f(a) - \epsilon < f(x)$. *Upper semi-continuity* at a is defined analogously.

Examples. 1. A function is continuous iff it is both upper and lower semi-continuous.

2. $A \subset X$ is open iff its characteristic function is l.s.c. Analogously, closed sets are characterized by having u.s.c. characteristic functions.

3. The pointwise min of a finite set of l.s.c functions is l.s.c. The pointwise max of a finite set of u.s.c functions is u.s.c.

4. Denote by $\mathcal{R}$ the set of continuous, rectifiable paths $c : [a, b] \to R^n$, with the sup metric. Then the arc length function $L : \mathcal{R} \to R$ is lower semi-continuous. (**Exercise 1.**)

Proposition 1. Let $g, h : M \to R$ be l.s.c resp. u.s.c functions on a C^k manifold M, with $h(p) < g(p), \forall p \in M$. Then there exists a C^k function $f : M \to R$, so that $h < f < g$ on M.

Proof. For each $p \in M$, let $a_p = (1/2)(g(p) + h(p))$, so $h(p) < a_p < g(p)$. Thus for some neighborhood V_p, $h(q) < a_p < g(q)$ for $q \in V_p$. This defines an open cover $\mathcal{C} = (V_p)_{p \in M}$ of M. Consider a C^k partition of unity $(\varphi_p)_{p \in M}$ subordinate to $\mathcal{C}$. Then $f = \sum_{p \in M} \varphi_p a_p$ is the sought-after function: $\varphi_p(q) = 0$ if $q \notin V_p$, and $h(q) < a_p < g(q)$ for $q \in V_p$, hence:

$$h(q) = \sum_p \varphi_p(q) h(q) < f(q) = \sum_p \varphi_p(q) a_p < g(q) = \sum_p \varphi_p(q) g(q).$$

Corollary 1. Let $\mathcal{C} = (C_\alpha)_{\alpha \in A}$ be a locally finite cover of a C^k manifold M, $(a_\alpha)_{\alpha \in A}$ a family of positive real numbers indexed by the same set A. Then there exists a C^k function $f : M \to R$ so that $0 < f(x) < a_\alpha$ if $x \in C_\alpha$.

Proof. We might as well assume the C_α are closed, since their closures also define a locally finite cover. Then let $g : M \to R$ be defined as $g(p) = \inf\{a_\alpha; p \in C_\alpha\}$. We *claim* g is lower semi-continuous. If so, Prop. 1 gives a C^k function f on M so that $0 < f < g$ on M, and in particular for $p \in C_\alpha$: $0 < f(p) < g(p) \leq a_\alpha$.

In fact, g is l.s.c in a curious way: any $p \in M$ has a neighborhood V_p so that $q \in V_p \Rightarrow g(q) \geq g(p)$: every point is a local min! To see this, observe that for each p there exists V_p intersecting only finitely many $C_{\alpha_1}, \dots C_{\alpha_r}$. Since these sets are closed, shrinking V_p if needed we may assume each V_p only intersects the C_{α_i} containing p.

In other words, if $q \in V_p$ and $q \in C_\alpha$, then $p \in C_\alpha$. Thus, for $q \in V_p$:

$$g(q) = \inf\{a_\alpha; q \in C_\alpha\} \geq \inf\{a_\alpha; p \in C_\alpha\} = g(p).$$

Corollary 2. Let M be a C^k manifold, $g : M \to R^n$ a continuous function. Given any positive continuous function $\epsilon : M \to R^+$, there exists a C^k function $f : M \to R^n$ so that $|f(x) - g(x)| < \epsilon(x)$ on M.

Proof. It suffices to consider the case $n = 1$, and then this follows from Prop. 1, which gives $f : M \to R$ of class C^k so that:

$$g(x) - \epsilon(x) < f(x) < g(x) + \epsilon(x), \quad \forall x \in M.$$

2. Function spaces. Let X be a topological space, Y a metric space. Denote by $W^0(X,Y)$ the set of continuous maps $f : X \to Y$, endowed with the topology for which the basic neighborhoods of $f \in W^0(X,Y)$ are the sets $W^0(f,\epsilon)$, where $\epsilon : X \to R^+$ is a positive continuous function on M and

$$W^0(f,\epsilon) = \{g \in W^0(X,Y); d(f(x), g(x)) < \epsilon(x), \forall x \in X\}.$$

As ϵ varies over all continuous positive functions on X, these sets define a local basis at f for a topology on X (check this.) This is called the C^0 *Whitney topology* (or 'fine topology') on the space of continuous maps.

If X is not compact, $W^0(X,Y)$ is not metrizable, since none of its points admits a countable basis of neighborhoods (*check this; what if X is compact?*) Still, we use the shorthand notation $d(f,g) < \epsilon$ to mean $d(f(x), g(x)) < \epsilon(x)$ for all $x \in X$.

An alternative way to obtain a basis of neighborhoods of $f \in W^0(X,Y)$ is to consider the sets $W(f,U)$, where U is an open set in $X \times Y$ containing the graph of f, and $W(f,U) = \{g \in W^0(X,Y); Graph(g) \subset U\}$.

Exercise 2. Verify that these two systems of neighborhoods of f (varying ϵ or varying U) define equivalent bases at f for a topology.

Corollary 2 says exactly that the C^k functions from M to R^n are dense in $W^0(M; R^n)$, if M is a manifold of class C^k. (Later we'll see this is also true for C^k maps from M to a manifold N.)

We denote by $C^0(X,Y)$ the space of continuous maps from X to Y, with the topology of uniform convergence on compact sets. Recall a basis of neighborhoods of $f \in C^0(X,Y)$ is given by the sets:

$$V(f,K,\delta) = \{g \in C^0(X,Y); d(f(x),g(x)) < \delta, \forall x \in K\}.$$

Of course the identity map $i : W^0(X,Y) \to C^0(X,Y)$ is continuous: the Whitney topology is finer than the u.o.c. topology.

If M is a differentiable manifold, $C^0(M,Y)$ is metrizable (*why?*) If, furthermore, the metric space Y has a countable basis, the same holds for $C^0(X,Y)$.

When X is compact, any positive continuous function $\epsilon : X \to R^+$ attains its minimum. Thus the identity in the other direction $j : C^0(X,Y) \to W^0(X,Y)$ is also continuous (*check this*): the two topologies are equivalent, and metrizable (via the sup metric.)

Evidently when M is a C^k manifold, the set of C^k maps from M to R^n is dense in $C^0(M;R^n)$, since the Whitney topology is finer.

The C^1 Whitney topology. Let M,N be differentiable manifolds of class C^k ($k \geq 1$). We assume the existence of an embedding $\Phi : N \to R^n$ of class C^k, for some n. (It will be seen later that this results in no loss of generality.) Indeed to simplify the notation we'll just assume N is a surface oof class C^k in R^n. Fix a Riemannian metric on M, of class C^{k-1} (that is, at least of class C^0.)

We denote by $W^1(M,N)$ the space of C^1 maps from M to N, with the topology in which a basis of neighborhoods of $f \in W^1(M,N)$ is given by sets of the form (where $\epsilon : M \to R^+$ is a continuous positive function on M):

$$W^1(f,\epsilon) = \{g \in W^1(M,N); |f(p)-g(p)| < \epsilon(p), \quad |df(p)-dg(p)| < \epsilon(p), \quad \forall p \in M\}.$$

Here we consider $T_{f(p)}N, T_{g(p)}N$ as subspaces of R^n, so we may regard $df(p), dg(p) \in \mathcal{L}(T_pM, R^n)$, and take in this space of linear maps the norm defined by the Riemannian norm on M and the usual norm on R^n.

Exercise 3. Show the space $W^1(M,N)$ is Hausdorff.

We'll show in the next section that the topology of $W^1(M,N)$ is independent of the choices of the embedding Φ and the Riemannian metric on M.

In general $W^1(M,N)$ is not metrizable; but we'll write $||f-g||_1 < \epsilon$ to mean $|f(p)-g(p)| < \epsilon(p)$ and $|df(p)-dg(p)| < \epsilon(p)$, for all $p \in M$.

On the space of C^1 maps from M to N a different topology is given by C^1-uniform convergence on compact subsets of M; we'll denote this topological space by $C^1(M,N)$, The basic neighborhoods of $f \in C^1(M,N)$ are the sets $V^1(f,K,\delta)$, where $K \subset M$ is compact and δ is a positive real number:

$$V^1(f,K,\delta) = \{g \in C^1(M,N); |f(p)-g(p)| < \delta \text{ and } |df(p)-dg(p)| < \delta, \forall p \in K\}.$$

(we may abbreviate this by saying $||f-g||_{C^1(K)} < \delta$.)

The identity map $i : W^1(M,N) \to C^1(M,N)$ is continuous: the C^1 Whitney topology is finer than the topology of C^1 uniform convergence on compact sets (C^1-u.o.c.)

When M is compact, the two topologies are equivalent.

The space $C^1(M,N)$ is metrizable, with countable basis. (*Why?*)

3. Mapping properties of the Whitney topology. In this section we consider how the C^1 Whitney topology behaves under C^1 mappings of manifolds. There are two situations to be understood.

(A) M_1, M_2 are manifolds, N a surface in euclidean space, $\phi : M_1 \to M_2$ a C^1 mapping. Composition defines a map:

$$\phi^* : W^1(M_2,N) \to W^1(M_1,N), \quad \phi^*(f) = f \circ \phi.$$

It turns out that ϕ^* is not always continuous, but is always a homeomorphism if ϕ is a diffeomorphism.

(B) M is a manifold, N_1, N_2 are surfaces in euclidean space, $\phi : N_1 \to N_2$ a C^1 map. We have:

$$\phi_* : W^1(M,N_1) \to W^1(M,N_2), \quad \phi_*(f) = \phi \circ f.$$

We'll see that ϕ_* is continuous, and a homeomorphism if ϕ is a diffeomorphism.

Note that $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. In particular, if ϕ is a diffeomorphism ϕ^* and ϕ_* are bijections, with inverses $(\phi^*)^{-1} = (\phi^{-1})^*$ resp. $(\phi_*)^{-1} = (\phi^{-1})_*$.

Let's deal with (A) first.

Example. When M is not compact, we may find a continuous positive function $\epsilon : M \to R^+$ with $\inf\{\epsilon(p); p \in M\} = 0$. (*Do this.*) Then for any $f \in W^1(M,N)$, the basic neighborhood $W^1(f,\epsilon)$ contains no constant maps (other than possibly f): the constant maps form a discrete subset of $W^1(M,N)$.

It follows that the natural inclusion $c : N \to W^1(M,N)$ (which maps $q \in N$ to the constant map with value q) is discontinuous if M is not compact and $\dim(N) > 0$.

On the other hand, note that if we consider the single-point manifold a, $W^1(a,N)$ is naturally homeomorphic to N, via $f \mapsto f(a)$. Thus if M is non-compact and $dim(N) > 0$, the map $\phi : M \to a$ induces $\phi^* : W^1(a,N) \to W^1(M,N)$, which via the natural diffeomorphism $W^1(a,N) \sim N$ corresponds to the inclusion by constant maps : $N \to W^1(M,N)$, which is discontinuous.

The following proposition allows us to deal with the case of compact M_1 in (A).

Proposition 2. Let $\phi : M_1 \to M_2$ be a C^1 map. Given $K \subset M_1$ compact and $\eta > 0$, there exists $\delta > 0$ so that if $f, g \in W^1(M_2, N)$ and $||f - g||_1 < \delta$ in $\phi(K)$, then $||f \circ \phi - g \circ \phi||_1 < \eta$ in K.

Proof. Let $a \geq 1$ be a real number so that $A \geq \sup\{|d\phi(p)|; p \in K\}$, and set $\delta = \eta/A$. Then if $|f(q) - g(q)| < \delta$ in $\phi(K)$, we have $|f \circ \phi - g \circ \phi| < \delta \leq \eta$ in K, and also, for each $p \in K$:

$$|d(f \circ \phi)(p) - d(g \circ \phi)(p)| = |df(\phi(p))d\phi(p) - dg(\phi(p))d\phi(p)|$$

$$\leq |df(\phi(p)) - dg(\phi(p))||d\phi(p)| < \delta A \leq \eta,$$

as we wished to show.

Remark. Note that this shows $\phi^* : C^1(M_2, N) \to C^1(M_1, N)$ is continuous, for any C^1 map $\phi : M_1 \to M_2$.

Corollary 1. If M_1 is compact and $\phi : M_1 \to M_2$ is a C^1 map, then $\phi^* : W^1(M_2, N) \to W^1(M_1, N)$ is continuous.

It suffices to observe that any positive continuous function ϵ on M_1 has a positive infimum (say $\eta > 0$.) The proposition gives a positive constant $\delta > 0$ such that:

$$g \in W^1(f, \delta) \Rightarrow g \circ \phi \in W^1(f \circ \phi, \eta) \subset W^1(f \circ \phi, \epsilon).$$

We now refine the above argument to establish that ϕ^* is continuous if ϕ is a proper map.

Proposition 3. Let $\phi : M_1 \to M_2$ be a C^1 proper map. Then $\phi^* : W^1(M_2, N) \to W_1(M_1, N)$ is continuous.

Proof. We first observe that if $\eta : M_1 \to R^+$ is continuous and $\phi : M_1 \to M_2$ is a continuous proper map, we may find a positive continuous function $\delta : M_2 \to R^+$ so that $0 < \delta(\phi(p)) < \eta(p)$, for all $p \in M_1$.

Indeed let $M_2 = \bigcup_{\alpha \in A} K_\alpha$ be a locally finite cover of M_2 by compact sets K_α. Since ϕ is proper, for each $\alpha \in A$ the preimage $\phi^{-1}(K_\alpha)$ is compact, so $a_\alpha = \inf\{\eta(p); p \in \phi^{-1}(K_\alpha)\}$ is positive. (Unless $\phi^{-1}(K_\alpha) = \emptyset$, in which case we set $a_\alpha = 1$.) By Corollary 1 of Proposition 1, there exists $\delta : M_2 \to R^+$ continuous, so that $0 < \delta(q) < a_\alpha$, for all $q \in K_\alpha$. If $p \in M_1$, $\phi(p) \in K_\alpha$ for some $\alpha \in A$. So it follows $\delta(\phi(p)) < a_\alpha \leq \eta(p)$, as we wished.

Given $\epsilon : M_1 \to R^+$ continuous, let $\eta(p) = \epsilon(p)/(1 + |d\phi(p)|)$, and find $\delta : M_2 \to R^+$ continuous as in the previous paragraph, satisfying $0 < \delta \circ \phi < \eta \leq \epsilon$ on M_1.

We claim that if $f, g \in W^1(M_2, N)$ with $g \in W^1(f, \delta)$, then $g \circ \phi \in W^1(f \circ \phi, \epsilon)$. Indeed from $|g - f| < \delta$ in M_2 we have $|g \circ \phi - f \circ \phi| < \delta \circ \phi < \epsilon$ on M_1. And from $|dg - df| < \delta$ on M_2 follows:

$$|d(g \circ \phi) - d(f \circ \phi)| = |(dg \circ \phi)(d\phi) - (df \circ \phi)(d\phi)| \leq |(dg - df) \circ \phi||d\phi|$$

$$< |(dg - df) \circ \phi|(1 + |d\phi|) < \delta(1 + |d\phi|) < \epsilon.$$

This concludes the proof of proposition 3.

Corollary 1. If $\phi : M_1 \to M_2$ is a C^1 homeomorphism onto a closed subset of M_2, then $\phi^* : W^1(M_2, N) \to W^1(M_1, N)$ is continuous.

Indeed in this case ϕ is a proper map.

Corollary 2. The topology of $W^1(M, N)$ does not depend on the choice of Riemannian metric on M.

Let g, h be Riemannian metrics on M, and set $M_1 = (M, g), M_2 = (M, h)$. The identity map $i : M_1 \to M_2$ is a diffeomorphism, and hence $i^* : W^1(M_2, N) \to W^1(M_1, N)$ is a homeomorphism. But i^* is evidently the identity, so this says the topologies of $W^1(M_1, N)$ and $W^1(M_2, N)$ are the same.

Before turning to part (B) (properties of ϕ_*) we make a general observation about the Whitney topology.

Lemma. Let $\mathcal{C} = (\mathcal{K}_\alpha)_{\alpha \in \mathcal{A}}$ be a locally finite cover of the manifold M by compact sets K_α. A basis of neighborhoods at $f \in W^1(M, N)$ may be obtained by considering all families $\tilde{a} = (a_\alpha)_{\alpha \in A}$ of positive real numbers indexed by A, and setting, for each family $\tilde{a}$:

$$W^1(f, \tilde{a}) = \{g \in W^1(M, N); ||g - f||_{C^1(K_\alpha)} < a_\alpha, \forall \alpha \in A\}.$$

Proof. Let $W^1(f, \epsilon)$ be a basic neighborhood of f. Define $\tilde{a} = (a_\alpha)_{\alpha \in A}$ by setting a_α equal to the infimum of ϵ over K_α, a positive real number. Clearly $W^1(f, \tilde{a}) \subset W^1(f, \epsilon)$.

Conversely, given the family $\tilde{a}$, from corollary 1 of proposition 1 there exists $\epsilon : M \to R^+$ continuous, so that $0 < \epsilon(p) < a_\alpha$, for $p \in K_\alpha$. Thus $W^1(f, \epsilon) \subset .W^1(f, \tilde{a})$.

Proposition 4. If $\phi : N_1 \to N_2$ is a C^1 map, the map $\phi_* : W^1(M, N_1) \to W^1(M, N_2)$ (given by $\phi_*(f) = \phi \circ f$) is continuous.

Proof. Fix a countable, locally finite cover of M by compact sets, $M = \bigcup_{i=1}^{\infty} K_i$. Let $f \in W^1(M, N_1)$. To prove the continuity of ϕ_* at f, we must show that given a sequence $\tilde{b} = (b_i)$ of positive real numbers, we may find a sequence $\tilde{a}$ of positive real numbers so that, for all $i \geq 1$:

$$||g - f||_{C^1(K_i)} < a_i \Rightarrow ||\phi \circ g - \phi \circ f||_{C^1(K_i)} < b_i.$$

This will be done in two steps.

Step 1. For each $i \geq 1$, let $L_i \subset N_1$ be a compact neighborhood of $f(K_i)$. Then $a_i = dist[f(K_i), N_1 \backslash L_i]$ is a positive number, and $|f - g| < a_i$ in $K_i \Rightarrow g(K_i) \subset L_i$.

Since ϕ is uniformly continuous on L_i, we may pick smaller a_i, if necessary, to guarantee that if $|x - y| < a_i$ for x, y in L_i, then $|\phi(x) - \phi(y)| < b_i$.

And then from $|g - f| < a_i$ on K_i follows $|\phi \circ g - \phi \circ f| < b_i$ on K_i.

Step 2. Note that since $d\phi(p) \in \mathcal{L}(T_p N_1, T_{\phi(p)} N_2)$ and $d\phi(q) \in \mathcal{L}(T_q N_1, T_{\phi(q)} N_2)$, the expression $d\phi(p) - d\phi(q)$ is *meaningless.*

To circumvent this, recall that if we assume $M_1 \subset R^r, M_2 \subset R^s$, we may find a neighborhood V of M_1 in R^r and extension $\Phi : V \to M_2$ (of class C^1) of the map ϕ. Then $d\Phi(p) - d\Phi(q) \in \mathcal{L}(R^r, R^s)$ does make sense.

If $g \in W^1(M, N_1)$, we have:

$$|d(\phi \circ f) - d(\phi \circ g)| = |d(\Phi \circ f) - d(\Phi \circ g)| = |(d\Phi \circ f)(df) - (d\Phi \circ g)(dg)|$$

$$= |(d\Phi \circ f)(df) - (d\Phi \circ g)(df) + (d\Phi \circ g)(df) - (d\Phi \circ g)(dg)|$$

$$\leq |d\Phi \circ f - d\Phi \circ g||df| + |d\Phi \circ g||df - dg|.$$

And now we impose the final restrictions on the a_i.

Since $d\Phi : L_i \to \mathcal{L}(R^r, R^s)$ is uniformly continuous, we may assume:

$$x, y \in L_i, |x-y| < a_i \Rightarrow |d\Phi(x) - d\Phi(y)| \sup_{K_i} |df| < \frac{b_i}{2}.$$

We may also assume:

$$a_i \sup_{L_i} |d\Phi| < \frac{b_i}{2}.$$

Then if $g \in W^1(M, N_1)$ is such that $|g-f|_{C^1(K_i)} < a_i$, it follows from the above calculation that $|d(\phi \circ g) - d(\phi \circ f)| < b_i$ on K_i.

Corollary. The topology of $W^1(M,N)$ does not depend on the embedding of N in some euclidean space.

Proof. Let $\phi_1 : N \to R^r, \phi_2 : N \to R^s$ be C^1 embeddings of N. Set $N_1 = \phi_1(N), N_2 = \phi_2(N)$. The map $\phi = \phi_2 \circ \phi_1^{-1} : N_1 \to N_2$ is a C^1 diffeomorphism (*justify*), so $\phi_* W^1(M; N_1) \to W^1(M; N_2)$ is continuous, and indeed a homeomorphism, since it is bijective with $(\phi_*)^{-1} = (\phi^{-1})_*$.

4. Stability of certain classes of C^1 maps.

In this section we show that if a C^1 map is an immersion, a submersion, an embedding, a diffeomorphism or transversal to a closed submanifold, then this property is preserved under small perturbations fo the map. First, a preliminary result.

Lemma 1. Let $U \subset R^m$ open, $K \subset U$ compact. Let $f \in C^1(U, R^n)$ be an immersion in K (that is, if $x \in K$, $df(x) \in \mathcal{L}(R^m, R^n)$ has trivial kernel.) Then there exists $\eta > 0$ so that if $g \in C^1(U, R^n), ||g-f||_{C^1(K)} < \eta$, then the restriction $g_{|K}$ is an immersion.

Proof. Let $\mathcal{O} \subset \mathcal{L} = \mathcal{L}(R^m, R^n)$ denote the open set of injective linear maps. Then $df : U \to \mathcal{L}$ is continuous and maps K to $\mathcal{O}$. Since K is compact and $\mathcal{O}$ is open, $\eta = dist[df(K), \mathcal{L} \setminus \mathcal{O}] > 0$. And if $g \in C^1(U, R^n), ||g - f||_{C^1(K)} < \eta$, then $dg(K) \subset \mathcal{O}$, as we wished to show.

Proposition 5. The C^1 immersions define an open subset $Imm^1(M,N) \subset W^1(M,N)$. Likewise, C^1 submersions define an open subset $Sub^1(M,N) \subset Im^1(M,N)$.

Proof. Let $U \subset M$ (open) be the domain of a chart $X : U \to R^m$, let $K \subset U$ compact, $f \in C^1(M,N), N \subset R^n$ an immersion on K. By the lemma and proposition 2 (mapping invariance of W^1), there exists $\delta > 0$ so that if $g \in C^1(M,N)$, $||g-f||_{C^1(K)} < \delta$, then g is an immersion on K.

Let $M = \bigcup U_i$ be a locally finite, countable open cover of M, with U_i the domain of a chart for M and (Vi) with $V_i \subset \overline{V_i} \subset U_i$ also a locally finite open cover. Let $f \in W^1(M,N)$ be an immersion. We have a sequence $\tilde{a} = (a_i)$ of positive real numbers so that if $g \in W^1(M,N)$ and $||g - f||_{C^1(\overline{V_i})} < a_i$, then $G_{|V_I}$ is an immersion. Thus the neighborhood $W^1(f, \tilde{a})$ of f in $W^1(M,N)$ consists only of immersions.

Exercise 4. Use a similar argument to prove the statement for submersions.

Remark. Recall from the example in the preface that Prop. 5 is false for $C^1(M < n)$: we need the Whitney topology. And, of course, the set of immersions from M to N may be empty (for instance, if $\dim(M) > \dim(N)$.)

The following lemma is used to prove openness of C^1 embeddings.

Lemma 2. Let $U \subset R^m$ open, $K \subset U$ compact convex, $f : U \to R^n$ a C^1 map such that $f_{|K}$ is an embedding. Then there exists $\eta > 0$ so that any $g \in C^1(U, R^n)$ with $||g - f||_{C^1(K)} < \eta$ is an embedding in K.

Proof. From Lemma 1, we know there exists $\eta' > 0$ so that $||g - f||_{C^1(K)} < \eta' \Rightarrow g_K$ is an immersion. We also knowo there exist $c > 0, \delta > 0$ so that $|f(x) - f(y)| > c|x - y|$ for any $x \in K, y \in U$ with $|x - y| < \delta$. By compactness, there exists $d > 0$ so that $|f(x) - f(y)| > d$ if $(x, y) \in A = \{(x, y) \in K \times K; |x - y| \geq \delta\}$, a compact set.

Let $\eta = \min\{\eta', \frac{c}{2}, \frac{d}{3}\}$. We claim if $g \in C^1(U, R^n)$ with $||g - f||_{C^1(K)} < \eta$, then $g_{|K}$ is injective. Indeed, let $x, y \in K, x \neq y$; set $h = g - f$. Then $|h(x)| < \delta, |dh(x)| < \delta$, for all $x \in K$, By the mean value inequality (since K is convex) we have $|h(x) - h(y)| < \eta|x - y|$, for all $x, y \in K$. Note that:

$$(i)|g(x) - g(y)| \geq |f(x) - f(y) - |h(x) - h(y)|;$$

$$(ii)|g(x) - g(y)| \geq |h(x) - h(y)| - |h(x)| - |h(y)|.$$

We consider two cases. First, assume $0 < |x - y| < \delta$. Then from (i):

$$|g(x) - g(y)| \geq c|x - y| - \frac{c}{2}|x - y| > 0.$$

Now if $|x - y| \geq \delta$, then $(x, y) \in A$, and we have from (ii):

$$|g(x) - g(y)| \geq d - \frac{d}{3} - \frac{d}{3} > 0.$$

Thus g is an injective immersion on K, and since K is compact, $g_{|K}$ is an embedding, proving Lemma 2.

Proposition 6. The C^1 embeddings $f : M \to N$ define an open subset $Emb^1(M, N) \subset W^1(M, N)$.

Proof. Let $M = \bigcup U_i$ be a 'good cover' of M (definition above). With $\overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i$, use the cover $M = \bigcup \overline{V_i}$ to define the topology of $W^1(M, N)$.

Let $f \in W^1(M, N)$ be an embedding. From Proposition 2 (mapping invariance of W^1) and Lemma 2, we have for each $i \geq 1$ a positive a_i so that if $g \in W^1(M, N)$ and $||g - f||_{C^1(\overline{V_i})} < a_i$, then $g_{|\overline{V_i}}$ is an embedding. Since f is a homeomorphism from M to $f(M)$, we have $d_i = dist(f(\overline{W_i}), f(M \setminus V_i)) > 0$,

Choose the a_i so that $a_i < \frac{d_i}{3}$ and $\lim a_i = 0$, We claim $W^1(f, \tilde{a}) \subset Emb^1(M, N)$. Clearly $W^1(f, \tilde{a}) \subset Imm^1(M, N)$. We show that any $g \in W^1(f, \tilde{a})$ is injective.

Let $p, q \in M, p \neq q$, We have $p \in W_i$ for some $i \geq 1$. If $q \in V_i$, then $g(p) \neq g(q)$. If $q \in M \setminus V_i$, then $|f(p) - f(q)| \geq d_i$, and therefore:

$$|g(p) - g(q)| \geq |f(p) - f(q)| - |f(p) - g(p)| - |f(q) - g(q)| \geq d_i - \frac{d_i}{3} - \frac{d_i}{3} > 0,$$

showing that g is injective on M.

We just have to show that if f is a homeomorphism onto $f(M)$ (so that the $d_i > 0$ are defined, as above) and $g \in W^1(f, \tilde{a})$ with $\lim_i a_i = 0$, then g is a homeomorphism onto $g(M)$. Thus we take a sequence $p_n \in M$ with $g(p_n) \to g(p)$, and must show $p_n \to p$. Note that we may assume there is a subsequence (denoted p'_n) leaving every compact subset of M eventually. (Restricted to a compact subset, injective immersions are embeddings.)

Fix an $i \geq 1$ so that $p \in W_i$. Then for n sufficiently large $p'_n \in M \setminus V_i$, and thus $|f(p) - f(p'_n)| \geq d_i$. Also, since $|f - g| < a_j$ on $\overline{V_j}$ and $\lim_j a_j = 0$, we have $\lim_n |g(p'_n) - f(p'_n)| = 0$, so $\lim_n f(p'_n) = g(p)$. So we have:

$$d_i \leq \lim_n |f(p) - f(p'_n)| = |f(p) - g(p)|,$$

a contradiction, since $|f - g| < d/3$ in $\overline{V_i}$. This shows that, in fact, all the p_n have to lie in some compact subset of M.

Question. Suppose $f \in W^0(M, N)$ is *proper.* If $a_i \to 0$ and $g \in W^0(F, \tilde{a})$, does it follow that g is proper?

(The connection is that an injective immersion which is also a proper map is an embedding. The converse is not true; for example, the inclusion oof the open disk into R^2 is an embedding, but not proper.)

Proposition 7. The set of all diffeomorphisms $f : M \to N$ (onto N!) is an open subset $Diff^1(M,N) \subset W^1(M,N)$.

Proof. (a) First assume M and N are connected. Leet $M = \bigcup U_i$ be a 'good cover' of M (definition above.) Given $f \in W^1(M,N)$, it has a neighborhood $W^1(f,\tilde{a})$ consisting only if embeddings into M. We claim that if $g \in W^1(f,\tilde{a})$ and $a_i \to 0$, then g is surjective. Since g is an open map and N is connected, it suffices to show $g(M)$ is closed in N.

So consider a sequence $g(p_n) \to q \in N$. We want to show there exists $p \in M$ so that $g(p) = q$. If p_n has a convergent subsequence $p'_n \to p \in M$, then $\lim g(p'_n) = g(p)$, so $g(p) = q$.

The alternative is that no compact subset of M contains infinitely many p_n. Thus, letting z$i_n = \inf\{i; p_n \in \overline{V_i}\}$' we have $\lim_n i_n = \infty$, so $\lim_n a_{i_n} = 0$. Thus:

$$0 \leq \lim_n |f(p_n) - g(p_n)| \leq \lim_n a_{i_n} = 0,$$

and $\lim_n f(p_n) = q$. Since f is onto n, there exists $p \in M$ so that $f(p) = q$. So $f(p_n) \to f(p)$, and since f is.a hoomeomorphism this implies $p_n \to p$. So $g(p) = \lim g(p_n) = q$, showing g is surjective.

(b) Moving to the general case, let $M = \bigcup M_s, N = \bigcup N_s$ be the decompositions into connected components. For each s, let $f(M_s) = N_{\bar{s}}$ and pick $p_s \in M_s$, $q_s = f(p_s) \in N_{\bar{s}}$. Since $N_{\bar{s}}$ is open in N, we have $dist(q_s, N \setminus N_{\bar{s}}) = c_s > 0$. Thus if $g \in W^1(M,N)$ with $|g - f| < c_s$ on M_s, we have $g(M_s) \subset N_{\bar{s}}$.

Using a covering $M = \bigcup K_i$ by compact connected sets to define the topology of $W^1(M,N)$, each K_i is contained in a connected component M_s. Thus if we require the sequence $\tilde{a} = (a_0)$ satisfies $a_i < c_s$ whenever $K_i \subset c_s$, it follows that the maps $g \in W^1(f,\tilde{a})$ satisfy $g(M_s) \subset N_{\bar{s}}$. The proposition then follows from case (a).

Remark. The conclusion of proposition 6 is not true in the space $C^1(M,N)$. For example, the identity map of the open unit disk D in R^2 is not an interior point of the subset of $C^1(D,D)$ made up of diffeomorphisms, *(Check this).* The point is that in the Whitney topology we are free to consider functions $\epsilon : D \to R^+$ with $\epsilon(x) \to 0$ as x approaches a boundary point. Thus a small perturbation of a diffeomomorphism onto D (in the Whitney sense) is forced to be *onto* D.

Prior to considering transversality, we prove a lemma regarding the stability of regular values.

Lemma 3. Let $K \subset M$ be compact, $\lambda : M \to R^s$ a C^1 map for which $0 \in R^s$ is a regular value, Then there exists $\delta = \delta(K) > 0$ so that if $\mu : M \to R^s$ is a C^1 map with $||\mu - \lambda||_{C^1(K)} < \delta$, then 0 is a regular value of $\mu_{|K}$.

Proof. The set U of $p \in M$ so that $d\lambda(p) \in \mathcal{L}(TM_p, R^s)$ has rank s is open, and is a neighborhood of the preimage $\lambda^{-1}(0)$; f is a submersion in U. Thus we may find an open set A containing $K \cap \lambda^{-1}(0)$, so that $\overline{A} \subset U$ is compact, with $\lambda_{|\overline{A}}$ a submersion. Moreover, $\lambda(K \setminus A)$ is a compact subset of R^s not containing 0; thus $dist[\lambda(K \setminus A), 0] = a > 0$. In addition (by the stability of submersions), there exists $\delta_1 > 0$ so that if $\mu \in C^1(M, R^s)$ satisfies $||\mu - \lambda||_{C^1(\overline{A})} < \delta_1$, then $\mu_{|\overline{A}}$ is a submersion, that is, any $y \in R^s$ is a regular value of $\mu_{|\overline{A}}$.

On the other hand, $||\mu - \lambda||_{C^1(K)} < a$ implies $0 \notin \mu(K \setminus A)$. Thus, setting $\delta = \min\{\delta_1, a\}$, we see that $||\mu - \lambda||_{C^1(K)} < \delta$ implies 0 is a regular value of $\mu_{|K}$. This concludes the proof of Lemma 3.

Proposition 8. Let S be a closed submanifold of N. Then the set of C^1 mappings $f : M \to N$ which are transversal to S is open in $W^1(M, N)$.

Proof. Let $\mathcal{C}$ be a covering of S by domains W of charts for N, $y : W \to R^n$, so that $y(W \cap S) \subset \pi^{-1}(0)$, where $\pi : R^n \to R^s$ projects on the last s coordinates (s is the codimension of S in N.)

Let $f \in W^1(M, N)$ be transversal to S.

Since S is closed in N, we may cover M by open sets U so that either $f(U) \cap S = \emptyset$ or $f(U) \subset W$ for some $W \in \mathcal{C}$. Refining this covering, we obtain $M = \bigcup U_i$ locally finite, with charts $x_i : U_i \to R^m$ such that $x_i(U_i) = B(3)$, and so that, for a given i, either $f(U_i) \subset N \setminus S$, or $f(U_i) \subset W$, for some $W \in \mathcal{C}$. Use the covering $M = \bigcup \overline{V_i}, V_i = x_i^{-1}(B(2))$ to define the topology in $W^1(M, N)$.

Given $i \geq 1$, there are two possibilities. The first is that $f(U_i) \cap S = \emptyset$. Since $f(\overline{V_i})$ is compact and disjoint from the closed set S, we may choose $a_i > 0$ so that $||g - f||_{C^1(\overline{V_i})} < a_i$ implies $g(\overline{V_i}) \cap S = \emptyset$. Thus g is trivially transversal to S on $\overline{V_i}$.

The second possibility is that $f(U_i) \cap S \neq \emptyset$, so $f(U_i) \subset W$ for some $W \in \mathcal{C}$. Then since f is transversal to S, considering the chart $y : W \to R^n$ and the projection $\pi : R^n \to R^s$, we know that $0 \in R^s$ is a regular value of the map $\lambda = \pi \circ y \circ f : U_i \to R^s$. By lemma 3, we may find $\delta_i > 0$ so that $||\pi \circ y \circ g - \pi \circ y \circ f||_{C^1(\overline{V_i})} < \delta_i$ implies $0 \in R^s$ is a regular value of

$\pi \circ y \circ g$. But then, by virtue of Proposition 4, we may find $a_i > 0$ so that $||g - f||_{C^1(\overline{V_i})} < a_i$ implies $||\pi \circ y \circ g - \pi \circ y \circ f||_{C^1(\overline{V_i})} < \delta_i$, and thus that g is transversal to S on $\overline{V_i}$.

The sequence $\tilde{a} = (a_i)$ defined in this fashion defines a neighborhood $W^1(f, \tilde{a})$ of f consisting only of maps $g : M \to N$ transversal to S, concluding the proof.

Remark. If S is not closed in N, the set of C^1 maps from M to N that are transversal t oS may fail to be open in $W^1(M, N)$. For instance, let $M = R, N = R^2$, $S = \{(x, x^2); x > 0\}$, $f : R \to R^2$ given by $f(x) = (x, 0)$. f is trivially transversal to S (since $f(R)$ does not intersect S.)

Exercise 5. Show that arbitrarily close to f (in the W^1 topology) one may find $g : R \to R^2$ which is not transversal to f.

5. Approximations in class C^1.

Corollary 2 of proposition 1 is an approximation theorem in class C^0: given a continuous map $f : M \to R^n$ on a manifold of class C^k, and a positive continuous function $\epsilon : M \to R^n$, to find an approximation $g : M \to R^n$ of class C^k, so that $g \in W^0(f, \epsilon)$, we proceeded as follows: given a locally finite open cover $M = \bigcup V_i$, for each i find $p_i \in V_i$ so that $|f(p) - f(p_i)| < \epsilon(p), p \in V_i$. Then if $(\varphi_i)_i$ is a partition of unity (of class C^k) subordinate to (V_i), take the weighted average $g(p) = \sum_i \varphi_i(p) f(p_i)$. This g is of class C^k and satisfies $|g(p) - f(p)| < \epsilon(p))$ for all $p \in M$.

We'll need the notation: (i) $V_\eta(K) = \bigcup_{p \in K} B_\eta(p)$, the η-neighborhood of $K \subset R^m$ compact. (ii) For $f, g : U \to R^n$ of class C^r ($r \geq 0$), $\delta > 0$ and $K \subset U$ (with $U \subset R^m$ open, K compact) we say $||f - g||_{C^r(K)} < \delta$ if:

$$|d^j f(x) - d^j g(x)| < \delta \quad \forall x \in K \text{ for } j = 0, 1, \ldots, r.$$

Proposition 8. Given $f : U \to R^n$ of class C^r, $K \subset U$ and $\delta > 0$, there exists $g : R^m \to R^n$ of class C^∞ so that $||g - f||_{C^r(K)} < \delta$ ($0 \leq r < \infty$).

Proof. By the differentiable Tietze extension theorem, there exists $h : R^m \to R^n$ of class C^r, coinciding with f in $\overline{V_\eta K}$. If $eta >$ is sufficiently small, then for $j = 0, 1,\ dots, r$:

$$\sup\{|d^j h(x + y) - d^j h(x)|; x \in K, |y| \leq \eta\} < \delta.$$

Let $\phi : R^m \to R$ be.a nonnegative C^∞ function so that $\phi(y) = 0$ for $|y| \geq \eta$ and $\int \phi d^m y = 1$. Defined $g : R^m \to R^n$ as the average:

$$g(x) = \int \phi(y) h(x + y) d^m y = \int \phi(z - x) h(z) d^m z$$

(by change of variable). By the Leibniz rule, for each $j = 1, \ldots, r$:

$$d^j g(x) = \int \phi(y) d^j h(x+y) d^m y = (-1)^j \int d^j \phi(z-x) h(z) d^m z.$$

Since $\phi \in C^\infty$, we see that $h \in C^\infty$. And since $\int \phi = 1$, we see that for each $x \in K$ and each $j = 0, 1, \ldots, r$:

$$|d^j g(x) - d^j f(x)| = |\int \phi(y)[d^j h(x+y) - d^j h(x)] d^m y|$$

$$\leq \sup_{|y| \leq \eta} |d^j h(x+y) - d^j h(x)| \int \phi(y) d^m y < \delta,$$

as we wished to show.

This leads directly to the main local smoothing result for maps from R^m to R^n:

Lemma. Let $f : B^m(3) \to R^n$ be a C^1 map. Given $\delta > 0$, there exists $h : B^m(3) \to R^n$ so that:
(1) $h = f$ in $B^m(3) \setminus B^m(2)$;
(2) $||h - f||_{C^1} < \delta$ in $B^m(3)$;
(3) $h \in C^\infty$ in $B^m(1)$.
(4) h is at least as differentiable as f, on any subset of $B^m(3)$.

Proof. Let $\varphi \in C^\infty(R^m)$ be an auxiliary function, with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $\overline{B^m(1)}$ and $\varphi \equiv 0$ in $\overline{B^m(3)} \setminus B^m(2)$. Let A be any positive real number so that $1 + ||\varphi||_{C^1} \leq A$. From Proposition 8, there exists a C^∞ map $g : R^m \to R^n$ such that $||g - f||_{C^1(\overline{B^m(2)})} < \delta/2A$.

Define $h = (1-\varphi)f + \varphi g$ (so (4) is guaranteed to hold.). Then:
(1) $h = f$ in $B^m(3) \setminus B^m(2)$.
(2) $|h - f| = \varphi|g - f| \leq |g - f| \leq \delta/2A < \delta$ in $B^m(3)$ and $|dh - df| = |(g-f)d\varphi + \varphi(dg - df)| \leq 2||\varphi||_{C^1}||g - f||_{C^1} < \delta$.
(3) In $B^m(1)$, $h = g$, so h is C^∞. This concludes the proof.

Proposition 9. Let M^m be a manifold and $N \subset R^n$ a surface, both of class C^k. The C^k maps from M to N are a dense subset of $W^1(M, N)$,

Proof. Given $f : M \to N$ of class C^1, and $\epsilon : M \to R^+$ continuous, we must find $g : M \to N$ of class C^k, so that $g \in W^1(f, \epsilon)$.

Fix a covering $\mathcal{C}$ of N by domains of charts $y : Z \to R^n$, and then cover M by open sets U with compact closure, so that $f(\overline{U}) \subset Z$, for some $Z \in \mathcal{C}$.

This covering of M may be refined to a second one, countable, locally finite and consisting of domains of charts $x_i : U_i \to R^m$, so that $x_i(U_i) = B^m(3)$. Thus for each i there is a coordinate chart (for N), $y_i : Z_i \to R^n$, so that the compact set $f(\overline{U_i})$ is contained in Z_i. As usual, we let $V_i = x^{-1}(B^m(2)), W_i = x_i^{-1}(B^m(1))$ and assume the W_i cover M.

We now construct, inductively, a sequence of C^1 maps from M to N: $f_0 = f, f_1, \ldots$, satisfying the conditions:

(1) $f_1 = f_0$ on $M \setminus V_1$, $f_2 = f_1$ on $M \setminus V_2, \ldots$ $f_i = f_{i-1}$ on $M \setminus V_i$.

(2) f_i is of class C^k on $W_1 \cup \ldots \cup W_i$.

(3) $||f_i - f_{i-1}|| < \frac{c}{2^i}$ on M.

(4)$f_i(\overline{U_j}) \subset Z_j$ for all j.

Assuming $f_0 = f, f_1, \ldots, f_{i-1}$ defined, satisfying (1)-(4), we construct $f_i : M \to N$ of class C^1.

From propositions 2 and 4 (continuity of induced mappings on W^1), there exists $\delta > 0$ so that, if $\lambda, \mu : B^m(3) \to y_i(Z_i)$ are C^1 maps with $||\lambda - \mu||_{C^1} < \delta$ on $\overline{B^m(2)}$, then:

$$||y_i^{-1} \circ \lambda \circ x_i - y_i^{-1} \circ \mu \circ x_i||_{C^1} < \epsilon/2^i \text{ on } \overline{V_i}.$$

Set $\lambda = y_i \circ f_{i-1} \circ (x_i)^{-1} : B^m(3) \to y_i(Z_i) \subset R^n$. By the preceding lemma, there exists $\mu : B(3) \to R^n$ of class C^1, with $\mu = \lambda$ on $B^m(3) \backslash B^m(2)$, of class C^k where λ is C^k, with $||\mu - \lambda||_{C^1} < \delta$ in $B^m(3)$.

Define $f_i : M \to N$ setting $f_i = f_{i-1}$ on $M \setminus V_i$ and $f_i = y_i^{-1} \circ \mu \circ x_i$ on U_i. Conditions (1)-(3) are clearly satisfied by f_i, and we need to verify (4).

Only a finite number of sets $\overline{U_j}$ intersect $\overline{U_i}$, since $\overline{U_i}$ is compact and the cover $\{U_1, U_2, \ldots\}$ of M is locally finite. For each of those $\overline{U_j}$, the compact set $K_j = \lambda(x_i(\overline{U_i} \cap \overline{U_j}))$ is contained in the open set $A_j = y_i(Z_j \cap Z_i)$. Hence $\rho_j := dist(K_j, R^n \setminus A_j) > 0$. If we take care to require $\delta < \rho_j$ for all indices j such that $\overline{U_j} \cap \overline{U_i} \neq \emptyset$, then we'll have the condition $f_i(\overline{U_j}) \subset Z_i$ satisfied for all $j = 1, 2, 3, \ldots$.

This completes the inductive definition of the sequence f_i of C^1 maps satisfying (1)-(4), starting at $f_0 = f$.

Each $p \in M$ has an open neighborhood W_s intersecting only a finite number of the sets V_j. Let i be the highest index such that $W_s \cap V_i \neq \emptyset$. Then $i \geq s$, so $f\_i \in C^k$ on W_s. Also $j > i \Rightarrow W_s \subset M \setminus V_j$, and thus $f_i = f_{i+1} = f_{i+2} = \ldots$ at all points of W_s. Thus it makes sense to define $g : M \to N$ by setting $g(p) = \lim f_i(p)$ at each $p \in M$. Clearly $g \in C^k$m since each $p \in M$ has a neighborhood W_s on which g coincides with the C^k map f_i. Finally, for $x \in W_s$ we have:

$$|g-f|_{C^1}(x) = |f_i-f_0|_{C^1}(x) \le |f_i-f_{i-1}|_{C^1}(x)+\ldots+|f_1-f_0|_{C^1}(x) < \sum_{r=1}^{i} \frac{\epsilon(x)}{2^r} < \epsilon(x),$$

concluding the proof.

Proposition 10. Let M be a C^k manifold ($k \ge 1$), $f \in W^1(M, R^s)$ an embedding. Then in each neighborhood of f there exist embeddings $g : M \to R^s$ such that:

(i) g is at least as differentiable as f, everywhere on M.

(ii) $g(M)$ is a surface of class C^∞ in R^s.

Proof. We'll follow the scheme of the previous proof. Let $\mathcal{U} = \{U_1, U_2, \ldots)$ be a locally finite cover of M by domains of coordinate systems $x_i : U_i \to B^m(3)$ so that, setting $V_i = x_i^{-1}(B^m(2)), W_i = x_i^{-1}(B^m(3))$, the W_i still cover M. Given a positive continuous function $\epsilon : M \to R^+$, we may assume (from Prop. 6) $W^1(f, \epsilon)$ is a neighborhood of f consisting only of embeddings.

To find a map $g \in W^1(f, \epsilon)$ satisfying (i) and (ii) we'll construct, inductively, a sequence of maps $f_0 = f, f_1, \ldots, f_i, \ldots$ from M to R^s, satisfying:

(1) $f_i = f_{i-1}$ on $M \setminus V_i$;

(2) $f_i(W_1 \cup \ldots \cup W_i)$ is a C^∞ surface in R^s;

(3) $|f_i - f_{i-1}|_1(x) < \epsilon(x)/2^i$ on M;

(4) f_i is at least as differentiable as f_{i-1}, everywhere on M.

Assuming $f0 = f, f_1, \ldots f_{i-1}$ exist, we define f_i as follows. Let $a = \inf\{\epsilon(p)/2^i; p \in \overline{V_i}\}$. There exists $b > 0$ so that if $\lambda, \mu : B^m(3) \to R^s$ are C^1 maps with $||\lambda - \mu||_1 < b$ in $\overline{B^m(2)}$, then $||\lambda \circ x_i - \mu \circ x_i||_1 < a$ in $\overline{V_i}$. (Cf. Prop. 2.)

Let $\lambda = f_{i-1} \circ (x_i)^{-1} : B^m(3) \to R^s$. By the lemma preceding Proposition 8, there exists a map $\mu : B^m(3) \to R^s$ such that $\mu = \lambda$ on $B^m(3) \setminus B^m(2)$, $\mu \in C^\infty$ on $B^m(1)$, $||\mu - \lambda||_1 < b$ in $B^m(3)$ and μ is noo less differentiable than λ.

Now let $f_i = \mu \circ x_i$ in U_i, $f_i = f_{i-1}$ in $M \setminus V_i$. It is easy to check that conditions (1)–(4) are met,

Exercise 6. Fill in all the details in this proof outline.

Remark. We'll show in the next chapter that, given any C^1 manifold M, there exists an embedding $f : M \to R^s$ in some euclidean space R^s.

Proposition 9 implies this embedding may be taken so that $f(M)$ is a C^∞ surface. Then considering the smooth lcoal parametrizations $\varphi \in C^\infty$ of $f(M)$, the maps $\varphi^{-1} \circ f$ will constitute a maximal atlas for M, of class C^∞, contained in the original C^1 atlas of M.

6. Whitney's immersion theorem.

The goal of this section is to prove that the set of C^1 immersions from a C^1 manifold M^m to R^s is open and dense in $W^1(M; R^s)$, provided $s \geq 2m$.

Lemma 1. Given $f : M^m \to R^s$ of class C^1, let $X = \bigcup_{i=1}^\infty N_i$ be a countable union of surfaces in R^s, of codimension greater than m. Then for almost every $v \in R^s$, we have $[f(M) + v] \cap X = \emptyset$..

Proof. Saying the intersection is not empty (for a given v) is equivalent to saying that, $v \in R^s$ is in the union of the images of the maps $\varphi_i : M \times N_i \to R^s$, $\varphi_i(p, q) = q - f(p)$. Since $\dim(M) + \dim(N_i) < s$ for all i, the image of each φ_i has measure zero in R^s. And so does the union of their images, proving the claim.

Lemma 2. Suppose $s \geq 2m$. Let $f : B^m(3) \to R^s$ be a C^r map ($r \geq 1$). Given $\epsilon > 0$ (constant), there exists an immersion $g : B^m(3) \to R^s$, of class C^∞, so that $|g - f|_1 < \epsilon$ in $B^m(3)$.

Proof. By virtue of Proposition 9, we may assume $f \in C^\infty$. We search for g of the form $g(x) = f(x) + Ax$, where A is an $s \times m$ matrix of small norm. Then $dg = df + A$. The problem is finding A (of arbitrarily small norm) so that $df)x) + A$ has rank at least m, for each $x \in B^m(3)$.

Now, the matrices of rank $i < m$ make up a surface $N_i \subset R^{ms}$ of codimension $(m-i)(s-i)$ (why?). Since $s \geq 2m$ and $i \leq m-1$, ot follows that $(m-i)(s-i) \geq 1.[2m - (m-1)] = m + 1$. Thus each N_i has codimension $> m$ in R^{sm}. Since $df : B^m(3) \to \mathcal{L}(R^m, R^s) = R^{sm}$ is a smooth map, Lemma 1 implies that for almost every matrix $A \in R^{sm}$, $df(x) + A$ has rank m, for all $x \in B^m(3)$, that is: g is an immersion. Since a set of measure zero may not contain any neighborhood of $0 \in R^{sm}$, we may choose A with arbitrarily small norm; so $|g - f|_1$ is arbitrarily small in $B^m(3)$.