REVIEW PROBLEMS ON ORIENTATION, BROUWER DEGREE AND SINGULARITIES (Source: Guillemin-Pollack)

1. Let $f : X \to Y$ be a surjective local diffeomorphism of connected oriented manifolds. Show that if the differential $df(x) : T_xX \to T_{f(x)}Y$ preserves orientation at one point, then f preserves orientation globally.

2. Prove that every compact oriented hypersurface in euclidean space is orientable. (*Hint:* Jordan-Brouwer separation theorem.)

3. Let Z be a codimension one submanifold of the oriented manifold Y, endowed with a Riemannian metric. Prove that the following are equivalent:

(i) Z is orientable;

(ii) There exists smooth field of normal vectors $n(z)$ along Z in Y;

(iii) The normal bundle $N(Z) = \{(z, v) \in TY; v \in (T_zZ)^{\perp} \subset T_zY\}$ is trivial;

(iv) There exists a smooth function r defined in a neighborhood of Z in Y, so that $Z = r^{-1}(0)$ and $dr \neq 0$ at points of Z.

Hint. Do $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$ first. For $(iii) \Leftrightarrow (iv)$, consider the ϵ-neighborhood theorem (p.69) and the Tubular Neighborhood Theorem (p.76).

4. Let $M \subset R^K$ be a submanifold, V a vector field on M. Then V is a smooth map $V : M \to R^K, V(x) \in T_xM$, a subspace of R^K. And for all $x \in M$, $dV(x) \in \mathcal{L}(T_xM, R^K)$. Show that the image of the linear map $dV(x)$ is contained in the subspace T_xM of R^k.

Hint: This is clear if $M = R^m \times \{0_{K-m}\} \subset R^K$. Reduce to this case by taking local charts.

5. A vector field X on a manifold M corresponds to a section of the tangent bundle: $F : M \to TM, \pi \circ X = id_M$. So $F(p) = (p, X(p)) \in TM$.

(i) Show that F is an embedding, so that $F(M)$ is a submanifold of TM diffeomorphic to M. For the identically zero vector field, this submanifold is the *zero section* $\Sigma_0 \subset TM$.

(ii) A singularity $z \in M$ of a vector field X is called *nondegenerate* (or 'simple') if $dX(z)$ has rank $m = \dim M$. Show this is the case if and only if F is transversal to Σ_0 at $(z, 0) \in \Sigma_0$. (*Hint:* problem 17 of problem set I.)

6. Vector fields on R^k may be identified with maps from R^k to itself. In particular, the map $z \mapsto z^m$ on $\mathbb{C} = R^2$ defines a vector field in R^2 with a singularity at the origin. Check that this singularity has index m. Check that in the case of $z \mapsto \bar{z}^m$, the origin is a singularity of index $-m$.

7. Prove that the Euler characteristic of S^n is 2 if n is even, 0 if n is odd. (*Hint:* If n is even, define a south-pole/north-pole vector field with only two singularities, and compute their indices; this works in n is odd too, or recall there is a vector field without singularities in this case.)

8. Compute the Euler characteristic of a compact oriented surface of genus g (a sphere with g handles attached), by suitably modifying a triangulation of the sphere, and attaching a triangulation of the handles.