PROPER COVERING MAPS[1]

1. Closed maps, perfect maps, proper maps.

A continuous map $f : X \to Y$ is *closed* if it takes closed subsets of X to closed subsets of Y. A better characterization is "continuity of the preimage" (as a set-valued map) in the following sense:

Lemma. $f : X \to Y$ is closed if, and only if, given $y \in Y$ and an open neighborhood U of $f^{-1}(y)$ in X, there exists $V \subset Y$ open, $y \in V$, so that $p^{-1}(V) \subset U$.

Proof. If f is closed, $f(U^c)$ is closed and does not contain y, hence there is an open nbd. V of y s.t. $V \cap f(U^c) = \emptyset$, or $f^{-1}(V) \subset U$.

Conversely, if continuity of the preimage holds, let $F \subset X$ be closed. If $y \notin f(F)$, then the open set $U = X \setminus F$ contains $f^{-1}(y)$. Continuity of the preimage then implies $\exists\ V \subset Y$ nbd of y such that $f^{-1}(V) \subset U$, or $V \cap f(F) = \emptyset$. Thus $f(F)$ is closed in Y.

Definition. A continuous surjective map $f : X \to Y$ is *perfect* if it is closed, with compact fibers ($f^{-1}(y)$ is compact, $\forall y \in Y$). A continuous surjective map $f : X \to Y$ is *proper* if the preimage of a compact subset of Y is a compact subset of X.

Proposition. Perfect maps are proper. The converse is true, provided the topology of Y is Hausdorff and *compactly generated.*

Proof. Assume f is perfect, and let $K \subset Y$ compact. Let $\mathcal{U}$ be an open cover of $f^{-1}(K)$. Then if $y \in K$, we can find a finite subcover $\{U_1^y, \ldots, U_n^y\}$ of the compact set $f^{-1}(y) \subset f^{-1}(K)$ by finitely many sets of $\mathcal{U}$. Let $U_y \subset X$ be their union, an open neighborhood of $f^{-1}(y)$. Since f is closed, the 'continuity of the preimage' condition implies there exists an open neighborhood $V_y \subset Y$ of y, so that $f^{-1}(V_y) \subset U_y$. Now take a finite subcover $\{V_{y_1}, \ldots, V_{y_m}\}$ of the open cover $\{V_y\}_{y \in K}$ of the compact set $K \subset Y$. Then the union $U_{y_1} \cup \ldots \cup U_{y_m}$ is a finite union of subsets from $\mathcal{U}$, covering $f^{-1}(K)$.

Before turning to the converse, recall a topology on Y is said to be 'compactly generated' if a set $C \subset Y$ is closed in Y iff $C \cap K$ is closed, for any compact subset $K \subset Y$. This is the condition needed to guarantee the set of continuous maps from Y to a metric space Z is closed in the compact-open topology (see [Munkres], p. 283/284]). It is a weak condition: if Y is either locally compact or first countable, its topology is compactly generated.

[1] Source: E. Lima, *Fundamental group and covering spaces*, A.K.Peters (2003)

Assume then Y is Hausdorff and compactly generated, and let $f : X \to Y$ be continuous, surjective and proper; we want to show f is 'perfect'. That preimages of points are compact is clear (given Y Hausdorff), so we must show f is closed. Let $F \subset X$ be closed. To show $f(F)$ is closed in Y, it is enough to show $f(F) \cap K$ is closed, for each $K \subset Y$ compact. But $f^{-1}(K) \cap F$ is compact in X, and hence $f(f^{-1}(K) \cap F) = K \cap f(F)$ is compact in Y, in particular closed in Y.

Conclusion. For all practical purposes, for a continuous surjective map 'proper' is equivalent to 'closed, with compact fibers'. (Fiber=preimage of a point.)

Remark: If X and Y are separable metric, a continuous map is proper if, and only if, for any sequence (x_n) in X without convergent subsequences, the sequence $f(x_n)$ has no convergent subsequences in Y. (Good general topology exercise! Recall in this case 'compact' and 'sequentially compact' are equivalent.)

2. Proper covering maps.

Let X, Y be Hausdorff, $f : X \to Y$ continuous, surjective and a local homeomorphism. Well-known examples show this is not enough to conclude f is a covering map. A natural question is: what more does one need to add? It turns out adding 'proper' is enough. More precisely, we have:

Theorem. Let $f : X \to Y$ be a surjective local homeomorphism. Suppose Y is connected. Then the following are equivalent:

(1) There exists an $n \geq 1, n \in \mathbb{N}$, so that $card(f^{-1}(y)) = n$, for all $y \in Y$;

(2) f is a closed map with compact fibers (i.e. a 'perfect' map; for most spaces Y of interest: (2)$\Leftrightarrow$ f is a proper map.)

(3) f is a covering map with finite fibers.

Proof. (1)$\Rightarrow$ (2). We show f is closed, using the 'continuity of preimage' condition. Let $y \in Y$, $A \subset X$ an open neighborhood of $f^{-1}(y)$; we must find an open neighborhood $V \subset Y$ of y, so that $f^{-1}(V) \subset A$. Note that if $f^{-1}(y) = \{x_1, \ldots, x_n\}$, we may find neighborhoods $x_1 \in W_1, \ldots x_n \in W_n$ which are pairwise disjoint, and with $W_1 \cup \ldots \cup W_n \subset A$. Consider the neighborhood of $y \in Y$ (recalling f is an open map, being a local homeomorphism):

$$V = \bigcap_{i=1}^{n} f(W_i).$$

We claim this V works! (That is, $f^{-1}(V) \subset A$.) Let $U_i = W_i \cap f^{-1}(V)$, let $U \subset X$ (open) be the union of the U_i. Then $U \subset A$, so it's enough to show $f^{-1}(V) \subset U$. Let $z \in f^{-1}(V)$, so $w = f(z) \in V$. But $f^{-1}(w) = \{z_1, \ldots, z_n\}$ with $z_i \in W_i$ for each $i = 1, \ldots n$. Since the W_i are disjoint and $f^{-1}(w)$ contains exactly n points, z must be one of the z_i. Thus $z \in U_i$ for some i, so $z \in U$.

(2)⇒(3). We assume f closed; the fibers of f, being compact discrete subsets of X, are finite with the same number of elements (since Y is connected), call it $n \geq 1$. Given $y \in Y$, with $f^{-1}(y) = \{x_1, \ldots x_n\}$, we must find a neighborhood $V \subset Y$ of y evenly covered by f.

Let $x_1 \in W_1, \ldots, x_n \in W_n$ be disjoint open neighborhoods of the x_i in X, so that f is a homeomorphism over its image on each W_i. Let U be the union of the W_i, so $U \subset X$ is an open neighborhood of $f^{-1}(y)$. Let $V \subset Y$ be an open neighborhood of Y contained in the intersection $f(W_1) \cap \ldots f(W_n)$ and such that $f^{-1}(V) \subset U$. (Existence of V guaranteed by the "continuity of the preimage" condition, since f is assumed closed.)

We *claim* V is evenly covered by f. Indeed:

$$f^{-1}(V) = \sqcup_i (f^{-1}(V) \cap W_i) = \sqcup_i U_i, \quad U_i = f^{-1}(V) \cap W_i \quad \text{(disjoint union.)}$$

Since $U_i \subset W_i$, f maps U_i homeomorphically onto V, so V is evenly covered and f is a covering map.

(3)⇒(1) is clear, since the cardinality of fibers is locally constant for any covering map, and Y is assumed connected.

Corollary. If $f : X \to Y$ is a surjective local homeomorphism and X is *compact* (and therefore also Y is compact), then f is a covering map. (X, Y Hausdorff.)