LECTURE 1. Differentiable manifolds, differentiable maps

Def: topological m-manifold. Locally euclidean, Hausdorff, second-countable space. At each point we have a *local chart.* (U, h), where $h : U \to R^m$ is a topological embedding (homeomorphism onto its image) and $h(U)$ is open in R^m.

Def: differentiable structure. An *atlas of class* C^r ($r \geq 1$ or $r = \infty$) for a topological m-manifold M is a collection $\mathcal{U}$ of local charts (U, h) satisfying:

(i) The domains U of the charts in $\mathcal{U}$ define an open cover of M;

(ii) If two domains of charts $(U, h), (V, k)$ in $\mathcal{U}$ overlap $(U \cap V \neq \emptyset)$, then the transition map:

$$k \circ h^{-1} : h(U \cap V) \to k(U \cap V)$$

is a C^r diffeomorphism of open sets in R^m.

(iii) The atlas $\mathcal{U}$ is maximal for property (ii).

Def: differentiable map between differentiable manifolds. $f : M^m \to N^n$ (continuous) is *differentiable at* p if for any local charts $(U, h), (V, k)$ at p, resp. $f(p)$ with $f(U) \subset V$ the map $k \circ f \circ h^{-1} = F : h(U) \to k(V)$ is differentiable at $x_0 = h(p)$ (as a map from an open subset of R^m to an open subset of R^n.)

$f : M \to N$ (continuous) is *of class* C^r if for any charts $(U, h), (V, k)$ on M resp. N with $f(U) \subset V$, the composition $k \circ f \circ h^{-1} : h(U) \to k(V)$ is of class C^r (between open subsets of R^m, resp. R^n.)

$f : M \to N$ of class C^r is an *immersion* if, for any local charts $(U, h), (V, k)$ as above (for M, resp. N), the composition $F = k \circ f \circ h^{-1}$ satisfies $ker[dF(x)] = \{0\}$ for all $x \in h(U)$, where $dF(x) \in \mathcal{L}(R^m; R^n)$ is the differential of F. (In particular, $m \leq n$ necessarily.)

$f : M \to N$ is a *diffeomorphism* if it is a homeomorphism onto N and, for any $p \in M$ and local charts $(U, h), (V, h)$ at $p, f(p)$ (resp.) and $F = k \circ f \circ h^{-1}$, the differential $dF(x) \in \mathcal{L}(R^m, R^n)$, $x = h(p)$, is an isomorphism. (In particular, $m = n$.)

If a C^r map $f : M \to N$ is a diffeomorphism , the inverse f^{-1} is of class C^r.

LECTURE 2. Tangent space at a point.

Recall a subset $M \subset R^n$ is a *m-dimensional surface of class C^k* in R^n $(m \leq n)$ if for any $p \in M$ we may find an open nbd. of p, $W \subset R^n$, and an immersion of class C^k, $\phi : U \to R^n$ ($U \subset R^m$ open) which is a homeomorphism onto its image: $\phi(U) = W \cap M$.

For surfaces in R^n, the tangent space T_pM at $p \in M$ is geometrically defined as the set of all velocity vectors $\alpha'(0)$ at $t = 0$ of C^1 curves α on M with $\alpha(0) = p$.

Exercise 1. Show that $T_pM = d\phi(x_0)[R^m]$, where $\phi(x_0) = p$, and that this subspace of R^n is independent of the choice of local parametrization ϕ of M near p.

On a manifold, we use instead the idea of tangent space as the set of 'directional derivatives' of differentiable functions, at $p \in M$.

Def. Let M^m be a C^r manifold, C^r_M the vector space of real-valued C^r functions on M; let $p \in M$. A *tangent vector* at p is a linear map $X_p : C^r_M \to R$ satisfying:

$(\text{i})_p$: if $f, g \in C^r_M$ coincide in an open neighborhood of p, then $X_p(f) = X_p(g)$.

$(\text{ii})_p$: $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$ (Leibniz rule.)

It follows easily from these properties that $X_p(const) = 0, X_p(f) = X_p(f + const)$. (Show this.)

The space T_pM of tangent vectors at p is a vector space of dimension m: it admits a standard *basis* $(\partial_{x_i|p})_{i=1}^m$ associated to a local chart (U, h) at p such that $h(p) = 0$:

$$\partial_{x_i|p}(f) = \frac{\partial(f \circ h^{-1})}{\partial x_i}(0), \text{ where } h(p) = 0, i = 1, \ldots, m.$$

Indeed, if $X_p \in T_pM$ and $f \in C^r_M$ with $f(p) = 0$ (as we may assume), we compute using $(\text{ii})_p$ (see notes):

$$X_p(f) = \sum_i v^i \partial_{x_i|p}(f), \quad v^i = X_p(x^i),$$

where $x^i : U \to R$ is the i^{th} component of the C^r map $h : U \to R$.

Def. The *differential of a C^1 map $f : M^m \to N^n$* at $p \in M$ is the linear map $df(p) \in \mathcal{L}(T_pM; T_{f(p)}N)$ given by:

$$df(p)[X_p] = Y_{f(p)}, \quad Y_{f(p)}(g) = X_p(g \circ f), \quad g \in C^r_N.$$

Note that $g \circ f \in C^r_M$ (check in a local chart.)

Exercise 2. The linear map $Y_{f(p)} \in \mathcal{L}(C^r_N; R)$ satisfies the conditions (i),(ii) at $f(p)$, required of an element of $T_{f(p)}N$.

Reality check. If the manifold N is $\mathbb{R}$ (the real line), the tangent space at $t_0 \in \mathbb{R}$ is one-dimensional:

$$\partial_{t|t_0}(g) = \frac{dg}{dt}(t_0), \text{ for } g \in C^1(\mathbb{R}); \quad T_{t_0}N = \{c\partial_{t|t_0}; c \in \mathbb{R}\}.$$

We have an isomorphism:

$$\mathbb{I}_{t_0} : T_{t_0}N \to \mathbb{R}; \quad \mathbb{I}_{t_0}(Y_{t_0}) = Y_{t_0}(id_{\mathbb{R}}) = c, \text{ where } Y_{t_0}(g) = c\frac{dg}{dt}(t_0).$$

Now let $f : M \to N = \mathbb{R}$ be C^1. Then:

$$df(p)[X_p] = Y_{f(p)}, \text{ where } Y_{f(p)}(g) = X_p(g \circ f), \text{ for all } g \in C^1(\mathbb{R}).$$

In particular, if $g = id_{\mathbb{R}}$, we have $Y_{f(p)}(id_{\mathbb{R}}) = X_p(f)$.

Thus, under the isomorphism $\mathbb{I}_{f(p)}$, which identifies $T_{f(p)}N$ with $\mathbb{R}$, we have:

$$\mathbb{I}_{f(p)}(df(p)[X_p]) = (df(p)[X_p])(id_{\mathbb{R}}) = Y_{f(p)}(id_{\mathbb{R}}) = X_p(f).$$

Informally, this identifies $X_p(f) \sim df(p)[X_p]$: $X_p(f)$ is a 'directional derivative' of f, as expected.

LECTURE 3. Tangent bundle; locally trivial vector bundles

Def. The *tangent bundle* of a C^r manifold M^m is the set:

$$TM = \{(p, X_p) \in M \times \mathcal{L}(C^r_M; R); X_p \in T_pM\},$$

that is, X_p satisfies $(i)_p, (ii)_p$. We have the standard projection map $\pi : TM \to M, \pi(p, (X_p)) = p$.

(Intuitively, we think of elements of TM as 'vectors with basepoint'–only vectors with the same basepoint can be added.)

We endow TM with a topology by imposing two conditions:

(a) π is a continuous map: $\pi^{-1}(U)$ is open in TM if $U \subset M$ is an open set;

(b) Let (U, h) be a local chart for M, $h(p) = 0$. Define $\tilde{h} : \pi^{-1}(U) \to U \times R^m$ by

$$\tilde{h}(p, X_p) = (h(p), v),$$

where $X_p = \sum_i v^i \partial_{i|p}$, the components of X_p in the basis $\partial_{i|p}$ of T_pM associated with the chart h. We require $\tilde{h}$ to be a homeomorphism.

Exercise 3. Show that (a),(b) define a topology on TM making it a topological manifold (Hausdorff, second countable, locally euclidean.) Describe a local basis of neighborhoods at a point (p, X_p) of TM.

Def. Differentiable structure on TM.

Let $(U, \varphi), (V, \psi)$ be overlapping charts for M, $U \cap V \neq \emptyset$. So $F = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a C^r diffeomorphism. Then the associated charts for TM also overlap, with the transition diffeomorphism (of class C^{r-1}):

$$\tilde{F} = \tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times R^m \to \psi(U \cap V) \times R^m.$$

$$(x, \alpha) \mapsto (F(x), dF(x)[\alpha]).$$

Thus the C^r atlas on M induces a C^{r-1} atlas on TM.

The tangent bundle is a special case of an important class of manifolds.

Def. A *locally trivial vector bundle* with typical fiber $\mathbb{V}$ is a triple (E, M, π), where E and M are differentiable manifolds and $\pi : E \to M$ a differentiable map, satisfying:

(i) Any $p \in M$ admits a neighborhood $U \subset M$ and a diffeomorphism ('local trivialization') $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{V}$ commuting with the identity on U: $\pi(e) = p_1(\Phi_U(e))$, where $p_1 : U \times \mathbb{V} \to U$ is the first projection.

(ii) If two trivializing domains overlap ($U \cap W \neq \emptyset$), the transition map has the form:

$$\Phi_W \circ \Phi_U^{-1} : U \times \mathbb{V} \to W \times \mathbb{V}, \quad (x, v) \to (x, T(x)v),$$

where $T : U \cap W \to GL(\mathbb{V})$ is a differentiable map.

E is *trivial* if there is a global diffeomorphism $\Phi : E \to M \times \mathbb{V}$, $\pi(e) = p_1(\Phi(e))$, so that $\Phi_{|E_p} \in \mathcal{L}(E_p, \mathbb{V})$, $E_p = \pi^{-1}(p)$.

Def. A *section* of E is a differentiable map $X : M \to E$ such that $\pi \circ \sigma = id_M$, i.e. $X(p) \in E_p$. A *vector field* on M is a section of TM: $X(p) \in T_pM$.

Exercise 4: E is trivial $\Leftrightarrow \exists$ a differentiable basis of sections $X_1, \ldots, X_n$ ($n = \dim \mathbb{V}$), i.e. $\{X_i(p)\}$ is a basis of E_p, for all $p \in M$.

Example 1: TS^1 is trivial: $V(x, y) = (-y, x)$ is a nonvanishing tangent vector field onS^1.

Example 2: TS^2 is not trivial: there is no non-vanishing tangent vector field on S^2. This follows from the following facts:

(i) If $f, g, \in C(S^n, S^n)$ and $f(x) + g(x) \neq 0 \forall x \in S^n \subset R^{n+1}$, then f is homotopic to g.

(ii) In particular, if $f \in C(S^2, S^2)$ has no fixed points, f is homotopic to α, the antipodal map α.

(iii) The identity map of S^2 is not homotopic to the antipodal map (this is true for even-dimensional spheres). The proof uses the homotopy invariance of the degree of a map.

(iv) Suppose V were a nonvanishing vector field on S^2. Consider the map: $f(x) = \frac{x+V(x)}{||x+V(x)||}$. Then f is homotopic to the identity (replace V by $tV, t \in [0,1]$). On the other hand, f has no fixed points, hence is homotopic to α. Contradiction!

Exercise 5. Prove (i), (ii) and (iv) in detail. In fact, prove the more general version of (i): if f, g satisfy $|f(x) + g(x)| < 2 \ \forall x \in S^n$, then f is homotopic to g.

Exercise 6. (i) Generalizing Example 1, show that any odd-dimensional sphere admits a non-vanishing vector field. (*Hint.* Consider S^{2n+1} as the unit sphere in $\mathbb{C}^n$, complex n-dimensional space:

$$S^{2n+1} = \{(z_1, \ldots, z_n); \sum_j |z_i|^2 = 1, z_j = u_i + iv_j.\}.$$

(ii) Show that the 3-sphere is *parallelizable* (i.e., the tangent bundle TS^3 is trivial). *Hint:* Recall S^3 can be thought of as the set of unit quaternions (denote the quaternion algebra by $\mathbb{H}$). Use quaternion algebra and trial and error to find three l.i. tangent vector fields at a point $x + yi + zj + wk \in S^3 \subset \mathbb{H}$.

LECTURE 4. Submanifolds, immersions and embeddings.

Recall the *Implicit Function Theorem* in euclidean space:

Theorem. Let $U \subset R^m$ open, $f : U \to R^m$ be a C^r map ($r \geq 1$). Suppose $df(x) \in \mathcal{L}(R^m)$ is an isomorphism, for some $x \in U$. Then there exist open neighborhoods U of x, V of $f(x)$ in R^m, such that:

(i) f is a homeomorphism from U to V;

(ii) $f^{-1} : V \to U$ is of class C^r, with $df^{-1}(f(x)) = (df(x))^{-1} \in \mathcal{L}(R^m)$.

Thus if $df(x)$ is an isomorphism for all $x \in U$, then f is a C^r local diffeomorphism (a C^r local homeomorphism with C^r local inverses.)

Exercise 7. If $f : U \to R^m$, $U \subset R^m$ open, is a C^1 immersion, show that $V = f(U)$ is open in R^m, and that if f is injective, then f is a C^1 diffeomorphism from U onto V (that is, a homeomorphism of class C^1, with C^1 inverse.)

Here we recall:

Def. A C^r map $f : M \to N$ of C^r manifolds is an *immersion* if $df(p)$ has rank $m = \dim M$, for all $p \in M$ (in particular, $m \leq n$.)

Up to taking local charts on the domain and range, immersions have a very simple form:

Theorem: local form of immersions. Let $f : M \to N$ be a C^k map, let $p \in M$ be such that $df(p)$ has rank $m = \dim M$. Then there exist a local chart (V, φ) at p, a real vector space F of dimension $n - m$, a neighborhood Z of $f(p)$ in N with $f(V) \subset Z$ and a C^k diffeomorphism onto its image $\psi : Z \to R^m \times F$ so that:

$$\Phi = \psi \circ f \circ \varphi^{-1} : \varphi(V) \to R^m \times F \text{ satisfies } \Phi(x) = (x, 0) \quad \forall x \in \varphi(V).$$

Proof. (Outline.) We consider the euclidean case, since the manifolds statement is easily reduced to it. So $f : U \to N$, with $U \subset R^m$ open and N an n-dimensional real vector space. The idea is to reduce the result to the inverse function theorem. So let $E = df(p)[R^m] \subset N$ (an m-dimensional subspace) and let $F \subset N$ be any subspace complementary to E, so $N = E \oplus F$. Now define:

$$h : U \times F \to N, \quad h(x, y) = f(x) + y, \quad h(p, 0) = f(p),$$

$$dh(p, 0) \in \mathcal{L}(R^m \oplus F; E \oplus F) \text{ invertible: } dh(p, 0)[u \oplus v] = df(p)[u] \oplus v.$$

By the IFT, h is a diffeomorphism from a neighborhood $V \times W \subset U \times F$ of $(p, 0)$ ($p \in V, 0 \in W$) to a neighborhood $Z = h(V \times W) \subset N$ of $f(p)$. (In particular, $f(V) = h(V \times \{0\}) \subset Z$.) Let $\psi = h^{-1} : Z \to V \times W$. Then since $h(x, 0) = f(x)$, we have, for all $x \in V$:

$$\Phi(x) = (\psi \circ f)(x) = h^{-1}(f(x)) = (x, 0),$$

as claimed.

Exercise 8. Show that (with V as in the statement of the theorem) the restriction $f_{|V} : V \to f(V) \subset Z$ is injective, and an open map (i.e. an *embedding* of V into $Z \subset N$), where $f(V)$ has the induced topology from Z. (But note in general $f(V)$ is *not* open in $f(M)$, unless $f : M \to N$ is *globally* an embedding.)

Thus, informally, the topological content of the 'local form of immersions' is 'any immersion is locally an embedding'.

Def: Submanifold. Let $M \subset N$ be C^r manifolds, of dimension m, n resp. ($m \leq n$.) M is a *submanifold* of N if for each $p \in M$ one may find a chart (V, ψ) for N at p, $\psi : V \to R^m \times R^{n-m}$, such that $\psi(M \cap V) \subset R^m \times \{0_{n-m}\}$, and $(V \cap M, \psi_{|V \cap M})$ is a chart in the C^r atlas of M. In particular, M is a topological manifold, in the topology induced from N.

Exercise 9: A C^r surface in R^n (see definition in an earlier lecture) is a submanifold of R^n; and conversely, a C^r submanifold of R^n satisfies the definition of a C^r surface in R^n.

We have the following natural question: if $f : M \to N$ is an injective embedding, when is $f(M)$ a submanifold of N? Simple examples (embeddings from R to R^2) show this is not always the case.

Proposition. Let $f : M \to N$ be a C^k immersion. If $f : M \to f(M)$ is an open map (where $f(M)$ has the topology induced from N) then $f(M)$ is a submanifold of M.

This follows from the local form of immersions (see class notes for a proof.) And conversely, if $f : M \to N$ is a differentiable injective immersion and $f(M) \subset N$ is a submanifold, then f is an embedding (since f is then an open map onto $f(M)$, by a previous problem.)

As a corollary, if an injective differentiable immersion f is a topological embedding (i.e. a homeomorphism onto its image, with the subspace topology), then $f(M)$ is a submanifold of N. This is always the case if M is compact, or more generally if f is *proper*, since proper maps are closed (and hence also open, when bijective).

Exercise 10. Recall a continuous map is *proper* if the preimage of compact sets is compact. Show that if M, N are differentiable manifolds and $f : M \to N$ a differentiable map, then f proper $\Rightarrow$ f closed.

However, it is easy to give examples of embeddings with image a submanifold, and which are not proper maps (for instance, the inclusion map of $(-1, 1)$ into $\mathbb{R}$).

LECTURE 5. Regular values, submersions

Implicit function theorem. -euclidean space

Let $f : U \to R^n$ be a C^k map, $U \subset R^m$ open. Suppose $p \in U$ is such that $df(p) \in \mathcal{L}(R^m; R^n)$ is surjective (so $m \geq n$). Let $c = f(p) \in R^n, L_c = f^{-1}(c)$.

Then in a neighborhood of p, L_c is a C^k surface of dimension $m-n$, and $T_pL_c = ker(df(p))$.

Specifically, let $R^m = E \oplus F$ be a direct sum decomposition, where $df(p)_{|F} \in \mathcal{L}(F; R^n)$ is an isomorphism: so F is n-dimensional and E is any complement of F in R^m. Write $p = (p_E, p_F) \in E \times F$. Then there exist open neighborhoods $Z \subset R^m$ of p, $V \subset E$ of p_E, $W \subset R^n$ of $f(p)$, and a diffeomorphism $\varphi : V \times W \to Z$ such that $(f \circ \varphi)(x, w) = w$. (*local form of submersions.*)

Proof. Consider $h : E \times F \to E \times R^n$, $h(x, y) = (x, f(x, y))$. So $h(p) = (p_E, f(p))$ and it is easy to see the differential of h is an isomorphism. By the IFT, we find neighborhoods $Z \subset R^m$, $V \subset E$, $W \subset R^n$ of $p, p_E, f(p)$ (resp.) so that h is a diffeomorphism from Z to $V \times W$. Let $\varphi = h^{-1}$. Then $(x, w) = h(\varphi(x, w)) = (x, f(\varphi(x, w))$, so $(f \circ \varphi)(x, w) = w$.

In fact $\varphi(x, w) = (x, \varphi_2(x, w))$. So define $\zeta : V \to F$ by $\zeta(x) = \varphi_2(x, c)$, where $c = f(p)$. Then it is easy to see $L_c \cap Z = graph(\zeta)$: L_c in Z is the graph of a C^k function defined on $V \subset E$, hence an $(m-n)$-dimensional surface of class C^k.

On manifolds, we have the same statement, proved from the euclidean one via local charts.

Def. A C^k map $f : M \to N$ is a *submersion* if $df(p)$ is surjective, for all $p \in M$. From the above, one easily shows:

Prop. Let $f : M^m \to N^n$ be a C^k submersion of C^k manifolds. Let $c = f(p) \in N, L_c = f^{-1}(c)$. Then L_c is a C^k submanifold of M, of dimension $m-n$, with tangent space $T_pL_c = ker(df(p))$.

Exercise 11. Any submersion is an open map.

Example 1. Let $M_n = R^{n^2}$ be the space of $n \times n$ real matrices, $det : M \to R$ the determinant function. Then for any $0 \neq c \in R$, the level set $\{X \in M_n; det(X) = c\}$ is a surface in M_n, of codimension 1. In particular, the matrix group SL_n (corresponding to $c = 1$) is a codimension 1 submanifold of R^{n^2}. (It is enough to show any $c \neq 0$ is a regular value of det.)

Example 2. The orthogonal group $O(n) = \{X \in M_n; XX^T = I_n\}$ is a *compact* surface of codimension $n(n+1)/2$ in M_n. This follows from the fact the identity matrix I_n is a regular value of the map $X \mapsto XX^T$ from M_n to Sym_n, the linear space of symmetric $n \times n$ matrices.

Exercise 12. The set of $m \times n$ matrices of rank r is a submanifold of

R^{mn} of codimension $(m-r)(n-r)$. ([G-P, p. 27])

LECTURE 6. Transversality

Def. Map transversal to a submanifold. A C^r map $f : M \to N$ is *transversal* at $p \in M$ to a submanifold $S \subset N$ if:

$$df(p)[T_pM] + T_qS = T_qN, \quad q = f(p). \qquad \text{Notation: } f \pitchfork_p S.$$

f is transversal to S if this holds for all $p \in f^{-1}S$ (or if $f^{-1}(S) = \emptyset$). (Notation: $f \pitchfork S$.)

Remark: Clearly if f is a submersion, f is transversal to any submanifold of N.

Transversal submanifolds. Two submanifolds $S_1, S_2 \subset N$ are transversal ($S_1 \pitchfork S_2$) if:

$$T_qS_1 + T_qS_2 = T_qN, \quad \forall q \in S_1 \cap S_2.$$

Equivalently, the inclusion map $i : S_1 \to N$ is transversal to S_2 at q (or the other way around.)

Transversality may be characterized in terms of regular values.

Proposition. Let $f(p) = q \in S$, $\psi : V \to R^s \times R^{n-s}$ be a submanifold chart for N at q (meaning $\psi(S \cap V) \subset R^s \times \{0_{n-s}\}$.) Let $U \subset M$ be a neighborhood of p so that $f(U) \subset V$. Then:

$$f_{|U} \pitchfork S \Leftrightarrow 0_{n-s} \text{ is a regular value of } \pi \circ \psi \circ f_{|U},$$

where $\pi : R^s \times R^{n-s} \to R^{n-s}$ is projection onto the second factor.

Corollary. If $f \pitchfork S$, the preimage $M_S = f^{-1}(S)$ is a submanifold of M, with codimension $codim_M(M_S) = codim_N(S)$. The tangent space at $p \in M_S$ is the preimage under the differential:

$$T_pM_S = (df(p))^{-1}T_qS, \quad f(p) = q.$$

This follows from $T_pM_S = Ker(d(\pi \circ \psi \circ f)(p))$ and $T_qS = Ker(d(\pi \circ \psi)(q))$, via the chain rule and the 'linear algebra fact':

$$Ker(ST) = T^{-1}Ker(S), \quad T \in \mathcal{L}(E;F), S \in \mathcal{L}(F;W).$$

Specializing to transversal submanifolds $S_1, S_2 \subset N$:

$$codim_N(S_1 \cap S_2) = codim_N(S_1) + codim_N(S_2), \qquad T_q(S_1 \cap S_2) = T_qS_1 \cap T_qS_2, \quad q \in S_1 \cap S_2.$$

In particular if $\dim(S_1) + \dim(S_2) = \dim(N)$, we have $\dim(S_1 \cap S_2) = 0$: the intersection consists of isolated points in N.

Def. Transversality of maps. Let $f : M \to P, g : N \to P$ be differentiable maps, $p \in M, q \in N$ such that $f(p) = g(q) = r \in P$. Then f and g are *transversal* at p, q (denoted $f \pitchfork_{p,q} g$) if:

$$df(p)[T_pM] + dg(q)[T_qN] = T_rP.$$

We say $f \pitchfork g$ if this is true at all p, q with $f(p) = g(q)$.

Remark: If f is a submersion, clearly $f \pitchfork g$ for any g.

Proposition. Let $\Delta \subset P \times P$ be the diagonal submanifold. Given $f : M \to P, g : N \to P$, consider the 'cartesian prodict map' $f \times g : M \times N \to P \times P$, $(f \times g)(p, q) = (f(p), g(q))$. Then if $f(p) = f(q) = r$:

$$f \pitchfork_{p,q} g \Leftrightarrow f \times g \pitchfork_{(p,q)} \Delta.$$

This follows from another linear algebra fact. Let A, B be subspaces of a vector space E. Then:

$$A + B = E \Leftrightarrow A \times B + \Delta = E \times E.$$

(Recall $E \times E$ has the 'direct sum' vector space structure.) Here Δ is the diagonal subspace of $E \times E$.

Exercise 13.

(i) Prove this linear algebra fact.

(ii) Show how the proposition follows from it. (You may use 'obvious' facts about tangent spaces of cartesian product manifolds.)

In the 'calculus case' of two differentiable (say smooth) functions $f, g : \mathbb{R} \to \mathbb{R}$, $f \pitchfork g$ is not quite the same as transversality of their graphs Γ_f, Γ_g at their points of intersection (as submanifolds of $\mathbb{R}^2$.)

Exercise 14. Draw examples showing these two notions of transversality are not the same. Analyze how they are related: does either one imply the other?

Corollary. Let $f : M \to P, g : N \to P$ be transversal maps. Then the 'coincidence set'

$$Q = \{(p, q) \in M \times N; f(p) = g(q)\}$$

is a submanifold of $M \times N$, of codimension equal to the dimension of P. (Since $Q = (f \times g)^{-1}(\Delta)$).

Examples. (i) Any $f : M \to N$ is transversal to the identity $id_N : N \to N$ (since id_N is a submersion.) Thus the graph of f: $\Gamma_f = \{(p, q) \in M \times N; q = f(p)\}$ is a submanifold of $M \times N$, of dimension $\dim M$. (This also follows from the fact that the map from M to $M \times N$, $p \mapsto (p, f(p))$ is an embedding– why?)

(ii) If $f : M \to N$ is a *submersion*, the graph of the equivalence relation induced by f is a submanifold of $M \times M$:

$$G = \{(p, q) \in M \times M; f(p) = f(q)\}.$$

Its dimension is $2 \dim M - \dim N$. (Why is this true?)

Exercise 15. (From [G-P p.33]). Let V be a real vector space, $T \in \mathcal{L}(V)$, $\Delta \subset V \times V$ the diagonal subspace, $\Gamma_T \subset V \times V$ the graph subspace of T. Then:

$$\Gamma_T \pitchfork \Delta \Leftrightarrow 1 \text{ is not an eigenvalue of } T.$$

In this case, what is the dimension of the intersection subspace $\Gamma_T \cap \Delta$? What is its dimension if 1 is an eigenvalue of T?

Exercise 16. If M is a compact manifold and $f : M \to M$ is a *Lefschetz map*, then f has only finitely many fixed points ([G-P p.33]).

Exercise 17. Let X be a vector field on M (a section of the tangent bundle $\pi : TM \to M$.) A singularity of X is a point $p \in M$ such that $X(p) = 0$. A *simple singularity* is a singularity p at which $dX(p) \in \mathcal{L}(T_pM, T_{0_p}TM)$ has rank m. Show that all singularities of X are simple iff $X \pitchfork \Sigma_0$, where $\Sigma_0 = \{(p, 0_p); p \in M\}$ is the 'zero section' of TM. Show that simple singularities are isolated.

LECTURE 7. Paracompactness and differentiable partitions of unity

Definitions. A collection $\{A_\alpha\}$ of subsets of a space M is *locally finite* if for any $p \in M$ there exists an open neighborhood U intersecting only finitely many A_α.

A space M is *paracompact* if any open cover of M admits a locally finite refinement.

Theorems (see [Munkres 1, no. 41])

1. Every paracompact Hausdorff space is normal.
2. Closed subspaces of paracompact spaces are paracompact.

3. Every metrizable space is paracompact.

4. Every regular Lindelöf space (in particular every regular second-countable space) is paracompact.

Proposition 1. X locally compact Hausdorff, second countable (for instance, a topological manifold. Then:

(i) X is σ-compact: there exist precompact open subsets $G_i, i \geq 1$, so that:

$$X = \bigcup_{i\geq 1} G_i, \qquad \overline{G_i} \text{ compact }, \qquad \overline{G_i} \subset G_{i+1}.$$

(ii) X is paracompact.

Proof. Let $\{U_\alpha\}$ be an open cover. Choose a finite subcover of the open cover $\{U_\alpha \cap (G_{i+1} \setminus \overline{G_{i-2}})\}$ of the compact $\overline{G_i} \setminus G_{i-1}$. Then choose a finite subcover of the open cover $\{U_\alpha \cap G_3\}$ of $\overline{G_2}$. The collection of such open sets is a countable, locally finite refinement of $\{U_\alpha\}$, consisting of precompact open sets.

In fact, one can do more. For a chart $h : U \to B(3) \subset R^m$ with $h(U) = B(3)$ (the open ball at 0 of radius 3), let $W \subset V \subset U$ be the preimages under h of the open balls $B(1) \subset B(2) \subset B(3)$.

Proposition 2. Let $\mathcal{C}$ be an open cover of M. Then $\mathcal{C}$ has a countable, locally finite refinement $\{U_1, U_2, \ldots\}$, domains of charts $h_i : U_i \to B(3)$ (onto), so that the corresponding W_i define a locally finite cover of M.

This is proved like Prop. 1. Note this is trivial if M is compact.

Definition. A set of C^k functions $\{\varphi_\beta\}_{\beta\in B} : M \to [0,1]$ is a *partition of unity* subordinate to an open cover $\mathcal{A} = \{U_\alpha\}_{\alpha\in A}$ of M if:

(i) the family $\{support(\varphi_\beta)\}_{\beta\in B}$ is locally finite, covers M, and is a refinement of $\mathcal{A}$;

(ii) The sum $\sum_{\beta\in B}\varphi_\beta$ is identically 1 on M.

A partition of unity is *strictly subordinate* to the open cover $\mathcal{A} = \{U_\alpha\}_{\alpha\in A}$ of M if the indexing set is the same and $support(\varphi_\alpha) \subset U_\alpha$, for each $\alpha \in A$.

There are two main existence results:

Proposition 3. Given an open cover $\mathcal{A} = \{U_\alpha\}_{\alpha\in A}$ of M, there exists a C^k countable partition of unity $\{\psi_i\}_{i\geq 1}$ subordinate to $\mathcal{A}$, where the supports $support(\psi_i)$ are compact (and define a locally finite, countable cover of M refining $\mathcal{A}$).

Proposition 4. Let $\mathcal{A} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then there exists a C^k partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ strictly subordinate to $\mathcal{A}$: $support(\varphi_\alpha) \subset U_\alpha$.

Remark. In general one cannot simultaneously require a partition of unity to be strictly subordinate to a given cover, and to have compact supports (as simple examples show.)

LECTURE 8. Applications of partitions of unity.

Application 1: differentiable Urysohn lemma. Let F, G be disjoint closed subsets of the C^k manifold M. Then there exists a C^k function $f : M \to [0,1]$ which equals 0 in F and equals 1 in G.

This follows from the existence of a partition of unity strictly subordinate to the open cover $\{M \setminus F, M \setminus G\}$ of M.

Application 2: Closed sets are level sets of differentiable functions. Let $F \subset M$ be a closed subset of a C^k manifold. Then there exists a C^k function $f : M \to R_+$ such that $f^{-1}(0) = F$. (This is an application of the differentiable Urysohn lemma.)

CASE 1: $F = K$ compact, $M = R^m$.

We have $K = \bigcap_i V_i$, where the V_i are the points in R^m with distance to K less than $1/i$: a decreasing family of precompact open sets, contracting to K. Then let f_i be a C^∞ function in R^m which is zero on K, 1 on $M \setminus V_i$. The problem is f_i could also vanish at other points. But suppose we find constants $c_i > 0$ so that $f = \sum_i c_i f_i$ is a smooth function in R^m. Then if x is in the complement of K, x is in some V_i, so $f(x) > 0$.

Note that all derivatives $f_i^{(j)}$ (for $j > 0$) are continuous functions with compact support (since $f_i \equiv 1$ outside the compact set $\overline{V_i}$.) Thus, for each $i \geq 1, j \geq 0$, we may find $M_{ij} > 0$ so that $|f_i^{(j)}| < M_{ij}$ in R^m. (Set $M_{i0} = 1$ for all i.) Now for each $i \geq 1$ fixed, it is easy to find, inductively, constants $\alpha_{ij} > 0$ decreasing in j, $\alpha_{i(j+1)} \leq \alpha_{ij}$, with $\alpha_{ij} \leq \frac{1}{2^i M_{ij}}$. For each j fixed, we have the bound:

$$\sum_i \alpha_{ij} |f_i^{(j)}| < \sum_i \frac{1}{2^i},$$

thus the series on the left converges absolutely and uniformly in R^m and defines a continuous function for each fixed j, $\sum_i \alpha_{ij} f_i^{(j)}$.

Now let $c_i = \alpha_{ii}$, and note $c_i \leq \alpha_{ij}$ for $i > j$. Thus $\sum_i c_i f_i$ converges uniformly on R^m to a function $f \in C(R^m)$, and the series $\sum_i c_i f_i^{(j)}$ also converge uniformly in R^m. Thus for the derivatives of any order: $f^{(p)} = \sum_i c_i f_i^{(p)}$, showing $f = \sum_i c_i f_i$ defines a C^∞ function with the desired properties.

Remark. Note f takes the constant value $\sum c_i$ in the complement of $\overline{V_1}$. Dividing by this number, we may assume $f \equiv 1$ outside of a compact set.

Exercise 18. If C is any closed subset of R^k, there exists a submanifold X of R^{k+1} such that $X \cap R^k = C$ (where R^k is embedded in R^{k+1} as the vectors with last component zero.) [G-P, p.33] (This shows that, without the requirement of transversality, the intersection of two submanifolds can be arbitrarily bad.)

CASE 2 (general). Take a countable, locally finite open cover of M by domains U_i of local charts, $h_i : U_i \to B(3) \subset R^m$. Let $W_i \subset V_i \subset U_i$ as before (preimages under h_i of balls of smaller radii), and let $K_i = \overline{V_i} \cap F$. Note $F = \bigcup_i K_i$. From the first part (using the charts h_i), we find $f_i \in C^k(M)$ with $f_i^{-1}(0) = K_i$ and $f_i \equiv 1$ outside V_i.

Now define $f : M \to R$ as the product of all the V_i! Since the cover $\{V_i\}$ of M is locally finite, near each $p \in M$ only finitely many f_i will be different from 1, so f is well-defined and of class C^k. In addition, $f(p) = 0 \Leftrightarrow f_i(p) = 0$ for some $i \geq 1 \Leftrightarrow p \in K_i$ for some $i \geq 1 \Leftrightarrow p \in F$. This concludes the proof.

Application 3. Extension of differentiable functions defined on arbitrary subsets of a manifold.

Application 4. Differentiable Tietze extension theorem.

LECTURE 9. Embeddings in euclidean space.

Theorem 1. Let M be an m-dimensional *compact* manifold of class C^r, which can be covered by n domains of coordinate charts $U_1, \ldots, U_n$, $h_i : U_i \to B(3)$, the ball of radius 3 in R^m. Let $\phi \in C^\infty(R^m)$ be a smooth 'bump function': equal to 1 in $B(1)$, equal to 0 in the complement of $B(2)$. Let $\varphi_i = \phi \circ h_i$ in U_i, extended to zero outside of U_i (so $\varphi_i \in C^r(M)$.)

Consider the map $f : M \to R^{(m+1)n} = R \times \ldots \times R \times R^m \times \ldots \times R^m$ ($2n$ factors):

$$f(x) = (\varphi_1(x), \ldots, \varphi_n(x), \varphi_1 h_1(x), \ldots, \varphi_n h_n(x).).$$

Then f is an injective immersion (and therefore an embedding, since M is compact.)

To extend this result (and proof) to the general (noncompact) case, we need the covering dimension theory in the Appendix. It is used to prove the following:

Theorem 2. Let M be an m-dimensional topological manifold. Then M may be covered by $m+1$ domains of coordinate charts (U_i, h_i), $i = 1, \ldots, m+1$.

Proof (outline). Let $\mathcal{A}$ be a locally finite open covering of M by finitely many domains of coordinate charts (which may assumed to have bounded image in R^m–why?) Referring to the Appendix, let $\mathcal{B} = \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_m$ be the refinement of $\mathcal{A}$ given by the Corollary in the Appendix. $\mathcal{B}_i$ is a countable *disjoint* union of domains of coordinate charts:

$$\mathcal{B}_i = \{V_{i1}, V_{i2}, \ldots\}, \quad k_{ij} : V_{ij} \to B_{ij} \subset R^m,$$

where the $B_{ij} = k_{ij}(V_{ij})$ are bounded open sets in R^m, but not necessarily disjoint (for fixed i and varying j.) To correct this, we replace the k_{ij} by $\tilde{k}_{ij}$, so that:

$$\tilde{k}_{ij}(V_{ij}) \cap \tilde{k}_{il}(V_{il}) = \emptyset \text{ if } j \neq l.$$

Exercise 19. (i) Explain how the $\tilde{k}_{ij}$ are constructed. Then define coordinate charts $h_i : U_i \to R^m$, where U_i is the disjoint union of the V_{ij} over j and $i = 1, \ldots .m+1$, concluding the proof of Theorem 2.

(ii) Show how one can cover the two-dimensional torus by three domains of coordinate charts to R^2 (not necessarily connected.)

Exercise 20. (i) Let M be an m-dimensional manifold of class C^k. Define an injective C^k immersion:

$$f : M \to R \times \ldots \times R \times R^m \times \ldots \times R^m = R^{(m+1)^2}$$

($m+1$ factors of type R, $m+1$ of type R^m.) That is, show that the map you define is injective and an immersion.

(ii) If M is noncompact, show that this map is *proper* (hence f is an *embedding*).

Appendix to Lecture 9: Covering dimension of manifolds.

Definition. A topological space X has covering dimension at most m ($\dim X \leq m$) if any open cover of X admits an open refinement with order at most m (any point is in at most $m+1$ open sets of the refinement.)

(i) If $\dim X \leq m$ and $Y \subset X$ is a closed subspace, then $\dim Y \leq m$.

(ii) If $X \subset \mathbb{R}^m$ (compact), then $\dim X \leq m$.

(iii) If $X = Y_1 \cup \ldots \cup Y_p$, where each Y_i is a closed subspace of X, then $\dim Y_i \leq m \forall i \Rightarrow \dim X \leq m$.

(Proved in [Munkres 1, no. 50].)

Exercise 21(i). Prove (using these facts) that any compact topological m-manifold M satisfies $\dim M \leq m$.

In fact this is true for any topological m-manifold, but is harder to prove.

Remark. If $\mathcal{C},\mathcal{U}$ are open covers of X, the intersection $\mathcal{C}\cap\mathcal{U} = \{C\cap U; C \in \mathcal{C}, U \in \mathcal{U}\}$ is also an open cover of X, refining both $\mathcal{C}$ and $\mathcal{U}$. If $\mathcal{C}$ is an open cover of X and $Y \subset X$ is a closed subspace, then the *restriction* of $\mathcal{C}$ to Y is the open cover of Y:

$$\mathcal{C}_{|Y} = \{C \cap Y; C \in \mathcal{C}\}.$$

Theorem. Let M be a topological m-manifold. Then $\dim M \leq m$.

Proof. (Outline–cp [Munkres 2] or problem 8, p.316 in [Munkres 1].) Recall M is σ-compact in a strong sense:

$$M = \bigcup_{n\geq 1} K_n, \quad K_n \text{ compact }, K_n \subset int(K_{n+1}).$$

Denote by A_n the compact 'annular region': $A_n = K_n \setminus int(K_{n-1})$ $(n \geq 2)$, $A_1 = K_1$. Then M is also the union of the A_n.

Exercise 21 (ii). $\dim(A_n) \leq m$ for all $n \geq 1$. (This is like the previous problem.)

Given an open cover of M, consider a refinement $\mathcal{B}_0$ with the property: if $U \in \mathcal{B}_0$ intersects K_n, then $U \subset K_{n+1}$. Thus each U is contained in at most two of the A_n. Such a refinement can be obtained by intersecting the given cover with the open cover $\{int(K_{n+1}) \setminus K_{n-1}\}_{n\geq 2}, int(K_2)$ of M.

Now inductively construct a sequence of refining covers of M:

$$\mathcal{B}_0 < \mathcal{B}_1 < \mathcal{B}_2 < \ldots, \qquad \mathcal{B}_n < \mathcal{B}_{n+1},$$

as follows:

Let $\mathcal{B}_1$ be a refinement of $\mathcal{B}_0$, so that the restriction of $\mathcal{B}_1$ to K_1 has order at most m (possible due to (i)–(iii) above.)

Now suppose $\mathcal{B}_n$ is a refinement of $\mathcal{B}_{n-1}$, whose restriction to K_n has order $\leq m$. And let $\mathcal{C}$ be a refinement of $\mathcal{B}_n$ whose restriction to the compact annular region A_{n+1} has order $\leq m$ (from the previous problem.)

The idea is to construct an open cover $\mathcal{B}_{n+1}$ of M refining $\mathcal{B}_n$, whose restriction to K_{n+1} has order $\leq m$, by 'splicing' $\mathcal{B}_n$ and $\mathcal{C}$, as follows. $\mathcal{B}_{n+1}$ consists of the following open sets:

(a) Open sets of $\mathcal{B}_n$ intersecting K_{n-1} (and hence contained in K_n.)
(b) Open sets of $\mathcal{C}$ which do not intersect K_n.
(c) Open sets of the form

$$U' = \bigcup\{V \in \mathcal{C}; V \in \mathcal{C}_n \text{ and } f(V) = U\},$$

where $U \in \mathcal{B}_n$, $\mathcal{C}_n$ is the set of $V \in \mathcal{C}$ intersecting K_n, but not K_{n-1}; and $f : \mathcal{C}_n \to \mathcal{B}_n$ is a fixed 'choice function' assigning to each such V an open set in $\mathcal{B}_n$ containing it (which exists since $\mathcal{C}$ refines $\mathcal{B}_n$).

Note that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$, since $\mathcal{C}$ does, and any set U' of type (c) is contained in a set of $\mathcal{B}_n$ (namely U).

Exercise 22. Prove the *claim:* $\mathcal{B}_{n+1}$ restricted to K_{n+1} has order $\leq m$.

Hint: Note $K_{n+1} = K_n \cup A_{n+1}$. We must show that any $p \in K_{n+1}$ is in at most $m+1$ open sets of $\mathcal{B}_{n+1}$. There are two cases to consider:

(i) $p \in A_{n+1}$. Then any open set of $\mathcal{B}_{n+1}$ containing p is of type (b) (a set in $\mathcal{C}$) or type (c) (a union of sets in $\mathcal{C}$). And $\mathcal{C}$ restricted to A_{n+1} has order $\leq m$.

(ii) $p \in K_n$. Then any open set of $\mathcal{B}_{n+1}$ containing p is of type (a) (a set in $\mathcal{B}_n$) or type (c) (a union U' of sets of $\mathcal{C}$, contained in $U \in \mathcal{B}_n$). We know that $\mathcal{B}_n$ restricted to K_n has order $\leq m$, but something still needs to be said to conclude the proof of the *claim.*

Now let $\mathcal{B}$ be the collection of open sets which are in all $\mathcal{B}_n$ for n sufficiently large. (In particular $\mathcal{B}$ refines each $\mathcal{B}_n$.) *Claim:* $\mathcal{B}$ is an open cover of M, with order $\leq m$.

The claim follows from the following two facts:

(i) If $B \in \mathcal{B}_N$ intersects K_{N-1} then $B \in \mathcal{B}_n$ for all $n \geq N$ (from the inductive construction), thus $B \in \mathcal{B}$. Such B cover K_{N-1}, so $\mathcal{B}$ covers M.

(ii) Let $x \in M$, $B_1, \ldots, B_p$ open sets in $\mathcal{B}$ containing x. Then one may find $N \in \mathbb{N}$ so that all $B_i \in \mathcal{B}_N$. Taking a larger N if needed, we may assume $x \in K_N$. Since $\mathcal{B}_N$ restricted to K_N has order $\leq m$, it follows that $p \leq m+1$. Thus any finite collection of open sets in $\mathcal{B}$ with a common point x has at most $m+1$ elements. This shows $\mathcal{B}$ has order $\leq m$.

Corollary. Let $\mathcal{A}$ be an open covering of M. There exists a locally finite open refinement $\mathcal{B}$, union of $\mathcal{B}_0, \ldots, \mathcal{B}_m$ (each $\mathcal{B}_i$ is a collection of open sets) so that the sets in each $\mathcal{B}_i$ are *disjoint.*

Proof outline (see [Munkres 2].) Let $\mathcal{C} = \{U_j\}$ be a refinement of $\mathcal{A}$ given by the result $\dim M \leq m$, and $\{V_j\}$ be a locally finite open cover such that $V_j \subset U_j \forall j$ (from paracompactness.) Let $\{\varphi_j\}$ be a partition of unity strictly dominated by $\{V_j\}$.

Given $n \in \{0, \ldots, m\}$, define $\mathcal{B}_n$ as follows: for each set of $n+1$ positive integers $i_0 < i_1 < \ldots < i_n$, define:

$$W(i_0, i_1, \ldots, i_n) = \{x \in M; \forall i \in \mathbb{N} \backslash \{i_0, \ldots, i_n\} \varphi_i(x) < \min\{\varphi_{i_0}(x), \ldots, \varphi_{i_n}(x)\}\}.$$

Let $\mathcal{B}_n$ be the collection of all such open sets W. It is not hard to show:
(i) The sets in $\mathcal{B}_n$ are all disjoint;
(ii) The union $\mathcal{B} = \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_m$ covers M.
(iii) $\mathcal{B}$ is locally finite.

LECTURE 10. Riemannian metrics

Definition. A *Riemannian metric* g on a differentiable manifold M is an assignment of an inner product (positive-definite bilinear form) on each tangent space T_pM. Notation $\langle v, w \rangle_p$ or $\langle v, w \rangle_g$, for $v, w \in T_pM$.

To make sense of differentiability, consider a local chart $h : U \to R^m$ at p ($h(p) = 0$) and the associated basis (∂_i) of T_pM. We have functions:

$$g_{ij} : U \to R, \quad g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p.$$

Let $\mathcal{S}_+(R^m) \subset \mathcal{L}(R^n)$ denote the set of positive-definite symmetric linear transformations, an open convex cone in the space $\mathcal{S}(R^m)$ of all symmetriic linear transformations. (Recall 'symmetric' means $\langle Tv, w \rangle_0 = \langle v, Tw \rangle_0$, for $v, w \in R^m$ and denoting the standard inner product ('dot product') in R^m with the subscript 0.) We have the map from U to $\mathcal{S}_+$:

$$g^h : U \to \mathcal{S}_+, \quad g^h(p)(v, w) = \langle dh^{-1}(0)[v], dh^{-1}(0)[w] \rangle_p., \quad v, w \in R^m.$$

We say g is of class C^k if, for each chart (U, h), the functions $g_{ij} \in C^k(U)$; equivalently, if $g^h : U \to \mathcal{S}_+$ is of class C^k.

Example. If (N, h) is a Riemannian manifold (a C^k manifold with a C^k Riemannian metric h), and $f : M \to N$ is a C^{k+1} *immersion*, we may induce a metric $g = f^*h$ ('pullback metric') on M by setting:

$$\langle v, w \rangle_g = \langle dh(p)v, dh(p)w \rangle_h, \quad v, w \in T_pM.$$

In particular, if M is an m-dimensional surface in R^n, the standard inner product in R^n induces in each subspace $T_pM \subset R^n$ (by restriction) an inner product on M.

In general, any C^k manifold carries a C^{k-1} Riemannian metric: cover M by countably many domains U_i of local charts (a loc. finite cover), introduce a metric in each U_i via local charts (pulling back from R^m) , then define a metric globally on M using a subordinate partition of unity.

1. Norm of the differential. Recall two natural ways to define a norm in a space of linear transformations:

(i) If E, F are normed vector spaces (finite-dimensional, over R), define the norm on $\mathcal{L}(E; F)$:

$$||T|| = \sup\{||Tv||_F; ||v||_E = 1\}.$$

Then clearly $||Tv||_F \leq ||T|| ||v||_E$. The problem with this norm is that the function $T \mapsto ||T||$ is not differentiable on $\mathcal{L}(E; F) \setminus \{0\}$ (as simple examples show.) It is only continuous in $\mathcal{L}(E; F)$.

(ii) If E, F are inner product spaces and $T \in \mathcal{L}(E; F)$, the adjoint $T^* \in \mathcal{L}(F; E)$ is defined by:

$$\langle Tv, w\rangle_F = \langle v, T^*w\rangle_E; \quad v \in E, w \in F.$$

We may define an inner product in $\mathcal{L}(E; F)$ via: $\langle T, S\rangle = tr(T^*S)$, where $T^*S \in \mathcal{L}(E)$ and tr means trace.

Then define the *Hilbert-Schmidt norm* on $\mathcal{L}(E; F)$ by:

$$||T||^2 = \langle T, T\rangle.$$

Introducing orthonormal bases in E, F we obtain matrices a_{ij}, b_{ij} for A, B and an isomorphism $\mathcal{L}(E; F) \sim R^{mn}$, with the natural inner product:

$$\langle A, B\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}, \quad m = \dim E, n = \dim F.$$

Then $||A||^2 = \sum_{i,j} a_{ij}^2$ is certainly smooth as a function of $A \in R^{mn}$, and $||A||$ is smooth away from the origin.

Proposition 1. Let $f : M^m \to N^n$ be a C^{k+1} map of Riemannian manifolds $(M, g), (N, g)$. Then the function $p \mapsto ||df(p)||^2$ is in $C^k(M)$ (where we use the Hilbert-Schmidt norm.)

Proof: It is enough to consider Riemannian metrics g resp. h in open sets $U \subset R^m, V \subset R^n$, and a map $f : U \to V$ of class C^{k+1}. Then we have C^k maps $G : U \to \mathcal{S}_+(R^m), H : V \to \mathcal{S}_+(R^n)$ so that:

$$\langle u, v\rangle_{g(p)} = \langle G(p)u, v\rangle_0, \quad p \in U, u, v \in R^m; \quad \langle w, z\rangle_{h(q)} = \langle H(q)w, z\rangle_0, \quad q \in V, w, z \in R^n.$$

Recall the Hilbert-Schmidt norm is given by:

$$||df(p)||^2 = tr(df(p)^* df(p)),$$

where * is the adjoint with respect to the metrics g, h. Let t denote the adjoint with respect to the euclidean inner products in R^m, R^n.

Exercise 23. Prove the linear algebra lemma:

$$G(p)df(p)^* = df(p)^t H(q), \quad q = f(p).$$

(State and prove the corresponding general lemma, for inner product spaces $(E, g), (F, h)$ and $T \in \mathcal{L}(E; F)$.)

Thus: $||df(p)||^2 = tr(G(p)^{-1} df(p)^t (H \circ f)(p) df(p))$, which is clearly C^k as a function of $p \in U$, proving the proposition.

2. The metric structure of a Riemannian manifold.

Let (M, g) be a connected Riemannian manifold (in particular, path connected.) Then given $p, q \in M$, there exists a piecewise C^1 path $\alpha : [0, 1] \to M$ from p to q (why?) with length $L[\alpha] = \int_0^1 ||a'(t)||dt$. Define:

$$d(p, q) = \inf\{L[\alpha]; \alpha : [0, 1] \to M \text{ is a p.w. } C^1 \text{ path from } p \text{ to } q\ \}.$$

Proposition 2. d defines a metric on M.
(The only condition that is not immediate is $p \neq q \Rightarrow d(p, q) > 0$.)

Proposition 3. (M, d) is homeomorphic to M with its original topology.

Proof. There are two things to show (see scanned notes):
(1) Any nbd. $V \subset M$ of $p \in M$ contains a d-ball with center p;
(2) Any metric ball $B_d(p, \epsilon)$ contains a coordinate neighborhood of p.

Remark. When d is a complete metric, we say M is a complete Riemannian manifold. In that case C^1 paths joining two points and achieving the infimum $d(p, q)$ (distance-minimizing curves) always exist, and (M, d) has the Heine-Borel property: closed bounded subsets are compact. The metric completion of a manifold is not always a manifold (e.g. remove the tangency point from a closed curve self-tangent at a point, bounding a connected region in the plane.)