MATH 562, spring 2021: FINAL EXAM (in class, 2h).

1. Let $p : S^1 \to S^1$, $p(z) = z^2$ $(z \in \mathbb{C}, |z| = 1)$.

(i) Prove p is a covering map, and that the covering is regular;

(ii) With $x_0 = \tilde{x}_0 = 1 \in \mathbb{C}$, identify (with proof) the subgroup $H(\tilde{x}_0) = p_*\pi_1(S^1, \tilde{x}_0)$ of $\pi_1(S^1, x_0)$, and the automorphism group of the covering.

2. (i) Define "properly discontinuous group action" (of a group G of homeomorphisms on a space Y).

(ii) Let $p : \tilde{X} \to X$ be a covering map. Prove that the automorphism group $G = Aut(\tilde{X}|X)$ of the covering acts properly discontinuously on $\tilde{X}$.

3. Let M be the compact orientable surface without boundary of genus 2 ('two-holed torus'); prove there exists a continuous map from M to S^1 which does not lift to $\mathbb{R}$ (under the standard covering map $p : \mathbb{R} \to S^1$).

4. Prove that any compact smooth hypersurface (without boundary) in $\mathbb{R}^n$ is orientable (*Hint:* Jordan-Brouwer separation theorem).

5. (i) Let $f : U \to R^k$ smooth, $U \subset R^k$ open, $x \in f^{-1}(z)$ a regular point, $g : \partial B \to R^k \setminus \{z\}$ the restriction of f to the boundary of a small ball with center x (containing no other points of $f^{-1}(z)$). Prove the winding number $W(g, z)$ equals ± 1, depending on whether f preserves (+1) or reverses (-1) orientation at x.

Hint: Assume $x = z = 0$. We have $f(x) = df(0)[x] + r(x)$ near 0, $r(x)/|x| \to 0$ as $x \to 0$. Define a homotopy between normalized f and normalized $df(0)$ on ∂B (replace $r(x)$ by $r(tx)/t$), then recall the group GL_k^+ of $k \times k$ matrices with positive determinant is connected.

(ii) $f : B \to R^k$ smooth, $B \subset R^k$ closed ball, z a regular value of f with no preimages on the boundary sphere ∂B, g the restriction of f to ∂B. Prove that the number of preimages of z in B (counted with orientation) equals the winding number $W(g, z)$.

6. Use the Hopf theorem on vector fields with isolated singularities on compact oriented manifolds to show the Euler characteristic of S^n is 2 if n is even, 0 if n is odd. (*Hint:* Describe a vector field on S^n with only two singularities, a source and a sink; and compute their indices.)