MATH 562, SPRING 2021–Take-home midterm

Sent: Friday, April 2, 7: 15PM. Solutions due: Monday, April 5, 7PM
Individual work. As references, you may use only the texts (Munkres 1&2, Gullemin-Pollack) and the material on the course web page.

Problem 1. Prove that there is no non-vanishing tangent vector field on S^2. Outline:

(i) If $f \in C^\infty(S^2, S^2)$ has no fixed points, f is homotopic to α, the antipodal map α.

(ii) The identity map of S^2 is not homotopic to the antipodal map.

(iii) Suppose V were a nonvanishing vector field on S^2. Consider the map: $f(x) = \frac{x+V(x)}{||x+V(x)||}$. Show f is homotopic to the identity. On the other hand, f has no fixed points (prove it), hence is homotopic to the antipodal map (justify.)

Problem 2. (i) Give an example of a local diffeomorphism $f : R^2 \to R^2$ which is not a diffeomorphism onto its image. (*Hint:* Consider the standard covering map $R \to S^1$.)

(ii) Prove that a local diffeomorphism $f : X \to Y$ which is also injective is a diffeomorphism from X onto an open subset of Y. (X, Y are smooth manifolds without boundary.)

Problem 3. Prove (directly, without citing the general result for $m \times n$ matrices) that the set of 2×2 matrices of rank 1 is a three-dimensional submanifold of $R^4 = M_{2\times 2}$. (*Hint:* Show that the determinant function is a submersion on the open set of nonzero 2×2 matrices.)

Problem 4. Let X, Y be smooth manifolds without boundary, with X compact. Prove that the set of C^1 submersions $f : X \to Y$ is open in the $C^1(X, Y)$ topology.

Problem 5. Let X, Y be smooth manifolds without boundary of the same dimension, with X compact and Y connected. Let $f : X \to Y$ be a smooth map such that $deg_2(f) \neq 0$. Prove that f is *onto.*

(ii) In the same setting as part (i) (without the hypothesis on mod 2 degree) prove that if Y is not compact (but X is), then $deg_2(f) = 0$.

Problem 6. Compute (with proof) the fundamental group $\pi_1(R^3 \setminus L)$, where L is the union of the coordinate axes. (Use any basepoint you like).