Tangent Spaces

Kelechi Akwataghibe

February 10, 2021

For surfaces in $\mathbb{R}^n$, the tangent space T_pM at $p \in M$ is geometrically defined as the set of all velocity vectors $\alpha'(0)$ at $t = 0$ of C^1 curves α on M with $\alpha(0) = p$.

Show that $T_pM = d\phi(x_0)[\mathbb{R}^m]$, where $\phi(x_0) = p$, and that this subspace of $\mathbb{R}^n$ is independent of the choice of local parametrization ϕ of M near p

Solution:

For the first part, we will show the following:

a For every curve in M, there is a velocity vector in T_pM

b Every vector in T_pM is the velocity vector of some curve in M

Proof. a Notice that the tangent space at a point on an interval $I \subset \mathbb{R}$ is $\mathbb{R}$. So, $\alpha'(t) : \mathbb{R} \to T_pM$

b To show that every vector in T_pM is the velocity vector of some curve in M. Fix $p \in M$ and $y \in T_pM$. Since $y \in T_pM$, $y = d\phi_{x_0}(w)$ for some $w \in \mathbb{R}^m$. Define $\widetilde{h} : \mathbb{R} \to \mathbb{R}^m$ by $t \mapsto (w_1t, \cdots, w_mt)$ and $h := \phi \circ \widetilde{h}$ Then, $dh_{x_0} = d\phi_{x_0} \circ \widetilde{h_{x_0}}$ and therefore, $dh_{x_0}(1) = d\phi_{x_0}(w)$

□

Part 2: This subspace is independent of the choice of local parametrization near p

Proof. {Adapted from Guillemin/ Pollack}
Let $\psi : V \to M$ be another choice of parametrization. [Given that $\phi : U \to M$ was the first choice]. By shrinking U and V, we may assume that $\phi(U) = \psi(V)$. The map $h = \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism. Hence, $\phi_{x_0} = \psi \circ h$ and differentiating, we get $d\phi_{x_0} = d\psi_{x_0} \circ dh_{x_0}$, and this implies that $d\phi_{x_0}[\mathbb{R}^m] \subset d\psi_{x_0}[\mathbb{R}^m]$. By switching ϕ and ψ, we have that $d\psi_{x_0}[\mathbb{R}^m] \subset d\phi_{x_0}[\mathbb{R}^m]$; therefore, $d\psi_{x_0}[\mathbb{R}^m] = d\phi_{x_0}[\mathbb{R}^m]$ and thus, the T_pM is well defined. □