Math 562, Topology

Set 2

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Proposition: Let $\Delta \in P \times P$ be the diagonal submanifold. Given $f : M \to P$ and $g : N \to P$, consider the cartesian product map $f \times g : M \times N \to P \times P$. Then if $f(p) = g(q) = r$:

$$f \pitchfork_{p,q} g \iff f \times g \pitchfork_{(p,q)} \Delta.$$

Lemma: Let A and B be subspaces of a vector space E. Then:

$$A + B = E \iff A \times B + \Delta = E \times E.$$

Problem 13: Prove the lemma and then use it to prove the proposition.

Proof of Lemma:

Let's first assume the LHS. Take $x, y \in E$ and let $a_x, a_y \in A$ and $b_x, b_y \in B$ such that $a_x + b_x = x$ and $a_y + b_y = y$. Since A and B are subspaces, $a_x - a_y \in A$ and $b_y - b_x \in B$. Moreover,

$$(a_x - a_y, b_y - b_x) + (a_y + b_x, a_y + b_x) = (a_x + b_x, a_y + b_y) = (x, y).$$

On the other hand, if we assume the RHS and take $x \in E$ non-zero with $a \in A$, $b \in B$ and $z \in E$ such that $(a, b) + (z, z) = (x, 0)$, it is clear that $z = -b \in B$ and therefore $x = a - b \in A + B$. ■

Proof of Proposition:

Let $A = df(p)[T_pM]$, $B = dg(q)[T_qN]$, and $E = T_rP$. But it is obvious(?) that

$$d(f \times g)(p, q)\left[T_{(p,q)}(M \times N)\right] = df(p)[T_pM] \times dg(q)[T_qN],$$

and $T_{(r,r)}P \times P = T_rP \times T_rP$, and finally that $T_{(r,r)}\Delta_P = \Delta_{T_rP}$. Since the LHS and RHS of the lemma now correspond to the definitions of transversality, the proposition follows trivially from the lemma. ■