MATH 562 - PROBLEM SET 2

PATRICK GILLESPIE

Exercise 15. Let V be a real vector space, $A \in \mathcal{L}(V)$, $\Delta \subset V \times V$ the diagonal subspace, $\Gamma_A \subset V \times V$ the graph subspace of A. Then:

$$\Gamma_A \pitchfork \Delta \iff 1 \text{ is not an eigenvalue of } A.$$

In this case, what is the dimension of the intersection subspace $\Gamma_A \cap \Delta$? What is its dimension if 1 is an eigenvalue of A?

Proof. First note that since A is a linear map, $dA_v = A$ for any $v \in V$. Hence if i is the inclusion of V onto Γ_A, defined by $i(v) = (v, A(v))$, then $T_{(v,A(v))}(\Gamma_A) = \operatorname{Im}(di_v) = \Gamma_A$. Similarly, the tangent space of $\Delta \subset V \times V$ at any point of Δ is an isomorphic copy of $\Delta \subset V \times V$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V.

So suppose 1 is not an eigenvalue of A. Then for all nonzero $v \in V$, $A(v) \neq v$, hence $\Gamma_A \cap \Delta = \mathbf{0}$. The dimension of $\Gamma_A \cap \Delta$ is 0. Then the vectors

$$\{(\mathbf{e}_1, A(\mathbf{e}_1)) \ldots, (e_n, A(\mathbf{e}_n)), (\mathbf{e}_1, \mathbf{e}_1), \ldots, (\mathbf{e}_n, \mathbf{e}_n)\}$$

are contained in $T_{\mathbf{0}}(\Gamma_A)) + T_{\mathbf{0}}(\Delta)$, and since $A(v) \neq v$ for all $v \in V$, this set of vectors spans $V \times V \cong T_{\mathbf{0}}(V \times V)$. Thus $\Gamma_T \pitchfork \Delta$.

Now if 1 is an eigenvalue of A, there exists some nonzero $v \in V$ such that $A(v) = v$. Writing $v = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_n \mathbf{e}_n$, we have that $\lambda_1 A(\mathbf{e}_1) + \cdots + \lambda_n A(\mathbf{e}_n) = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_n \mathbf{e}_n$, hence the vectors

$$\{(\mathbf{e}_1, A(\mathbf{e}_1)) \ldots, (e_n, A(\mathbf{e}_n)), (\mathbf{e}_1, \mathbf{e}_1), \ldots, (\mathbf{e}_n, \mathbf{e}_n)\}$$

(which still spans $T_{\mathbf{0}}(\Gamma_A)) + T_{\mathbf{0}}(\Delta)$) are not linearly independent. Therefore $T_{\mathbf{0}}(\Gamma_A)) + T_{\mathbf{0}}(\Delta) \neq T_{\mathbf{0}}(V \times V)$ and so Γ_A and Δ do not intersect transversely at $\mathbf{0}$. The dimension of $\Gamma_A \cap \Delta$ in this case is simply the dimension of the eigenspace of A associated with the eigenvalue of 1. $\square$

Date: February 26, 2021.