Problem Set 2

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Exercise 16. If M is a compact manifold and $f : M \to M$ is a Lefschetz map, then f has only finitely many fixed points.

Proof. Recall that a fixed point x of f is a *Lefschetz fixed point* if $\mathrm{d}f_x$ does not have 1 as an eigenvalue. A map f is called *Lefschetz* if all of its fixed points are Lefschetz fixed points.

Let x be a fixed point of f. By choosing a proper chart (U, ϕ), we may view f as a function $F : B \to B$, where B is a neighborhood of 0 in $\mathbb{R}^n$ for some n and 0 is the fixed point of F. By the Lefschetz condition, $\mathrm{d}F_0$ does not have 1 as an eigenvalue, so that the derivative of the function $G = F - \mathrm{id}$ at 0 has nonzero determinant. That is,

$$\det(\mathrm{d}G_0) = \det(\mathrm{d}F_0 - \mathrm{id}) \neq 0.$$

Therefore G is a local diffeomorphism by the inverse function theorem. Since G is a diffeomorphism in a neighborhood around 0, it is bijective in this neighborhood, in particular injective. Then if y were another fixed point in that neighborhood, $G(0) = 0$ and $G(y) = 0$, contradicting the injectivity of G. Therefore any fixed point of f is isolated.

But since M is compact and metrizable, it is limit point compact, so any infinite set has a limit point. If the set of fixed points of f were infinite, then there would be a fixed point which was not isolated. Hence f has finitely many fixed points. □