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Problem Set 1

Math 562
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Exercise 17

Let X be a vector field on M (a section of the tangent bundle $\pi : TM \to M$.) A *singularity* of X is a point $p \in M$ such that $X(p) = 0$. A *simple singularity* is a singularity p at which $dX(p) \in \mathcal{L}(T_pM, T_{0_p}TM)$ has rank m ($\dim M = m$). Show that all singularities of X are simple iff $X \pitchfork \Sigma_0$, where $\Sigma_0 = \{(p, 0_p); p \in M\}$ is the 'zero section' of TM. Show that simple singularities are isolated.

Proof :

(i) ($\Rightarrow$)
Suppose all singularities of X are simple.
Then rank $dX(p) = m$, for all singularity points $p \in M$.

Want to show $dX(p)[T_pM] + T_{0_p}\Sigma_0 = T_{(p,0_p)}TM \qquad (*)$

Now, $\dim dX(p)[T_pM] = m$, $\dim T_{0_p}\Sigma_0 = m$, and $\dim T_{(p,0_p)}TM = 2m$.

So, we only need to show that the sum in $(*)$ is a direct sum.
i.e. $dX(p)[T_pM] \cap T_{0_p}\Sigma_0 = \{0\}$
To show that, we want $dX(p)[v] \in T_{0_p}\Sigma_0 \implies v = 0$.

Since the claim is local, we take local coordinates for TM at $(p, 0_p)$ and identify:
$U \subset \mathbb{R}^m$, $\pi^{-1}(U) = TM|_U \sim U \times \mathbb{R}^m$ with coordinates (x, y).

$p \longmapsto x$

$$v_p \longmapsto \sum_{i=1}^{m} y^i \frac{\partial}{\partial x^i}\bigg|_p .$$

So, $\Sigma_0 \sim U \times \{0_m\} = \{(x, 0_m); x \in U\}$

$$dX(p)[v_p] \sim \left(x, \sum_j \frac{\partial a^i}{\partial x^j}\bigg|_x y^j;\ \ i = 1, \cdots, m\right) \in U \times \mathbb{R}^n, \qquad \text{where } X_p = \sum a^i(x) \frac{\partial}{\partial x^i}\bigg|_p$$

Thus $dX(p)[v] \in T_{(p,0_p)}\Sigma_0$ means $\displaystyle\sum_j \frac{\partial a^i}{\partial x^j}\bigg|_x y^j = 0, \quad i = 1, \cdots, m.$

Since, ker $dX(p) = \{0_p\}$, it follows that $(y^j) = 0_m$, or $v = 0$.

($\Leftarrow$) Now, suppose $X \pitchfork \Sigma_0$ i.e. $dX(p)[T_pM] + T_{0_p}\Sigma_0 = T_{(p,0_p)}TM$.
$\dim T_{(p,0_p)}TM = 2m$ and $\dim T_{0_p}\Sigma_0 = m$ implies that $\dim dX(p)[T_pM] = m$.

Which means rank $dX(p) = m$
So, all singularities of X are simple.

(ii) Simple singularities are isolated.
Claim: If p is a simple singularity, then $\exists$ a neighborhood U of p in which all singularities are simple.
Proof of claim: If p is a simple singularity, then rank $dX(p) = m$.
Then, since $X \pitchfork \Sigma_0$, 0_m is a regular value of $\pi \circ \varphi \circ X$, where

$$\varphi : V_{X(p)} \subset TM \to \mathbb{R}^m \times \mathbb{R}^m, \qquad \varphi(V \cap \Sigma_0) \subset \mathbb{R}^m \times \{0\}_m.$$

And $\exists p \in U \subset M$ open such that $d\left(\pi \circ \varphi \circ X|_U\right)(q)$ is onto, $q \in U$. i.e. $dX(q)$ has rank m.
Therefore, if q is a regular point, then q is simple.

Since 0_m is a regular value of $\pi \circ \varphi \circ X|_U$, $\left(\pi \circ \varphi \circ X|_U\right)^{-1}(0_m)$ is a sub-manifold of U with codim m.
Which means the set of simple singularities is a 'zero-dimensional' sub-manifold of U.
i.e. simple singularities are isolated.