Problem 18

If C is any closed subset of R^k, then there exists a submanifold X of R^{k+1} such that $X \cap R^k = C$.

Proof. Note that C, as an embedded subset of R^{k+1}, is a closed subset of R^{k+1}. Thus, we know from lecture that there exists a C^∞ function $f : R^k \mapsto R_+$ such that $f^{-1}(0) = C$. Define X to be the graph of f in R^{k+1} (that is, $X = \{(x, f(x)) : x \in R^k\}$). Observe that $X \cap R^k = C$ (where R^k is really $R^k \times \{0\} \subset R^{k+1}$). It remains to show that X is a submanifold of R^{k+1}.

Define the map $F : R^{k+1} \mapsto R$ by $F(x, x_{k+1}) = x_{k+1} - f(x)$. Note that F is class C^∞ since f is C^∞. So, dF is defined for all points in R^{k+1}. Calculating dF yields

$$dF = \begin{bmatrix} -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} & \cdots & -\frac{\partial f}{\partial x_k} & 1 \end{bmatrix}$$

Note that $dF : R^{k+1} \mapsto R$ is surjective for all $p \in R^{k+1}$. Indeed, for $y \in R$, we have

$$dF(p) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y \end{bmatrix} = y.$$

Therefore, F defines a C^∞ submersion from R^{k+1} to R. Recall the following proposition from the lecture:

Proposition 1. *Let $f : M^m \mapsto N^n$ be a C^k submersion of C^k manifolds. Let $c = f(p) \in N$, $L_c = f^{-1}(c)$. Then L_c is a C^k submanifold of M of dimension $m - n$.*

By this proposition, the set $F^{-1}(0)$ must be a submanifold of R^{k+1}. Now, observe that $X = F^{-1}(0)$. Thus, X is a submanifold of R^{k+1}.

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