1 Manifold Matrices

The set of $m \times n$ matrices of rank r (μ_r), is a submanifold of $\mathbb{R}^{mn}$ of codimension $(m-r)(n-r)$

Proof. Using the hint outlined in GP, we consider a set U of $m \times n$ matrices in the form :

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B is a non-singular $r \times r$ matrix.

Then, U is open in $\mathbb{R}^{mn}$. We first consider $L = U \cap \mu_r$

Let $A \in U$. We show that $A \in L$ iff $E - DB^{-1}C = 0$. This follows from by right-multiplying A with the non-singular matrix

$$rk(A) = rk\left(A.\begin{bmatrix} I & -B^{-1}C \\ 0 & I \end{bmatrix}\right) = rk\left(\begin{bmatrix} B & 0 \\ D & -DB^{-1}C + E \end{bmatrix}\right)$$

$\Leftarrow$ If $E - DB^{-1}C = 0$, and B is non-singular, rk(A) = r
$\Rightarrow$ If rk(A) = r, and B is nonsingular , then $E - DB^{-1}C = 0$ otherwise $E - DB^{-1}C$ would have a nonzero column, and $rk(A) > r$

Next, we parametrize $L = U \cap \mu_r$ via $\Phi : L_0 \to L$

$$L_0 = \{(B, C, D) \in \mathbb{R}^{r^2} \times \mathbb{R}^{r(n-r)} \times \mathbb{R}^{r(m-r)}; detA \neq 0\}$$

$$\Phi(B, C, D) = \begin{bmatrix} B & C \\ D & -DB^{-1}C \end{bmatrix} \text{ which is clearly a smooth immersion}$$

Noting that $\pi(A) = (B, C, D)$ satisfies $\pi \circ \Phi = id$, we conclude that Φ is homeomorphic onto its image L

Therefore, L is a smooth surface in $\mathbb{R}^{mn}$ of codimension $mn - (r^2 + r(n-r) + r(m-r)) = (m-r)(n-r)$.

If $A \in \mu_r$, we can permute its columns to get a matrix $PA \in U \cap \mu_r = L$ where $P \in U$ is a permutation matrix .

Now, if $P_1, \cdots, P_m$ are all the $m \times m$ permutation matrices, we can easily see that $P_j L \cap P_k L = \emptyset$ if $j \neq k$; since a matrix in the intersection will have rank ¿ r. Thus μ_r is the disjoint union:

$$\mu_r = \bigcupdot_{j=1}^{m!} P_j L; \text{ and therefore is a surface in } \mathbb{R}^{mn}$$

□