Math 562, Topology

Set 1

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Definition: The differential of a C^1 map $f : M^m \to N^n$ at $p \in M$ is the linear map $df(p) \in \mathcal{L}(T_pM; T_pN)$ given by:

$$df(p)[X_p] = Y_{f(p)},\ Y_{f(p)}(g) = X_p(g \circ f),\ g \in C^r_N.$$

Problem 2: Prove that the linear map $Y_{f(p)}$ is indeed in $T_{f(p)}N$.

Solution:

First notice that since f is C^1 and g is C^r, we have that for any charts (U, h) and (V, k) of M and N, the map $k \circ f \circ h^{-1}$ is C^1, while for any chart (W, l) of N the map $g \circ l^{-1}$ is C^r. Since we know that composition of maps between Euclidean spaces preserves continuity and differentiability, we can choose $l = k$ and get that the map

$$\left(g \circ k^{-1}\right) \circ \left(k \circ f \circ h^{-1}\right) = g \circ f \circ h^{-1}$$

is C^1 for any chart (U, h) of M, i.e., $g \circ f$ is C^1.

But then using the properties of $X_p \in T_pM$ we can get the following:

(i) If $g_1, g_2 \in C^r_N$ coincide in an open neighborhood of $f(p)$, then clearly $g_1 \circ f$ and $g_2 \circ f$ coincide in an open neighborhood of p (f is continuous). Therefore $X_p(g_1 \circ f) = X_p(g_2 \circ f)$, which by definition means that $Y_{f(p)}(g_1) = Y_{f(p)}(g_2)$.

(ii) For $g_1, g_2 \in C^r_N$, Leibniz rule in T_pM yields

$$X_p\left((g_1 \circ f)(g_2 \circ f)\right) = (g_1 \circ f)(p) X_p(g_2 \circ f) + (g_2 \circ f)(p) X_p(g_1 \circ f).$$

But this is the same as

$$Y_{f(p)}(g_1 g_2)) = g_1(f(p))\, Y_{f(p)}(g_2) + g_2(f(p))\, Y_{f(p)}(g_1).$$

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