Topology Homework

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Problem 20. Let M be an m dimensional manifold covered by $m+1$ coordinate charts $\{(U_i, h_i)\}$. We may, in the construction of these coordinate charts, arrange things so for each i, there is an open $W_i \subset \overline{W}_i \subset U_i$ and such that the W_i cover M. Let $\{\psi_i\}$ be a collection of bump functions with $\operatorname{supp}\psi_i \subset U_i$ such that $\psi_i \equiv 1$ on $\overline{W}_i$.

Define a map $\Phi : M \to \mathbb{R}^{(m+1)^2} = \mathbb{R}^N$ by

$$\Phi(p) = (\psi_1(p),\ \psi_2(p),\ \cdots,\ \psi_{m+1}(p),\ \psi_1(p)h_1(p),\ \psi_2(p)h_2(p),\ \cdots,\ \psi_{m+1}(p)h_{m+1}(p)),$$

where the $m+1$ functions $\psi_i h_i$ have been extended to M smoothly, since $\psi_i \equiv 0$ outside of U_i. We want to show that this is an embedding.

Φ *is injective:* If $p, q \in M$ are so $\Phi(p) = \Phi(q)$, then $\psi_i(p) = \psi_i(q)$ for every i. There exists some i_0 so $\psi_{i_0}(p) = \psi_{i_0}(q) = 1$. Then, since $\psi_{i_0}(p)h_{i_0}(p) = \psi_{i_0}(p)h_{i_0}(q)$, we have that $h_{i_0}(p) = h_{i_0}(q)$, so $p = q$ since h_{i_0} is a bijection.

Φ *is an immersion:* Let $p \in M$. There is a natural identification between $d_p\Phi$ and the product map of differentials

$$d_p\psi_1 \times \cdots \times d_p\psi_{m+1} \times d_p(\psi_1 h_1) \times \cdots \times d_p(\psi_{m+1}h_{m+1}).$$

Furthermore, there is some W_i containing p, so that $\psi_i \equiv 1$ on a sufficiently small neighborhood of p. In coordinates, the $(i+m+1)^{\text{th}}$ differential map above becomes

$$d_{h_i(p)}((\psi_i h_i) \circ h_i^{-1}) : \mathbb{R}^m \to \mathbb{R}^m$$

$$= d_{h_i(p)}(\mathrm{id}_{\mathbb{R}^m}),$$

which is the identity linear transformation, which has rank m. Thus, in local coordinates about p, the matrix representing $d_p\Phi$ has a rank m submatrix—so $d_p\Phi$ is an immersion.

Φ *is proper:* Note that all the spaces involved here are metrizable, so it suffices to use the sequential characterization of proper maps. Let $(x_n) \subset M$ be a sequence of points on our manifold which go to infinity (i.e., the sequence is eventually outside every compact subset of M). Our goal is to show that the sequence of images $\Phi(x_n) \to \infty$ in $\mathbb{R}^N$ as well. Suppose this were not the case—then there is a compact set containing a subsequence of $\Phi(x_n)$. In fact, by Heine-Borel, we may pass to another subsequence to find a sequence of elements $x_{n_k} = y_k \in M$ so that $y_k \to \infty$ in M and for which the sequence $\Phi(y_k) \to z$ for some $z \in \mathbb{R}^N$. Let $z = (z^1, z^2, \cdots, z^{m+1}, \mathbf{z}^1, \mathbf{z}^2, \cdots, \mathbf{z}^{m+1}) \in \mathbb{R}^N$. Then,

$$\psi_i(y_k) \to z^i$$

and

$$\psi_i(y_k)h_i(y_k) \to \mathbf{z}^i$$

as $k \to \infty$, for each $i = 1, 2, \cdots, m+1$. Now, since the sets $\overline{W}_i = \psi_i^{-1}(1)$ cover M, then

$$\sum_{i=1}^{m+1} \psi_i \geq 1$$

on the whole manifold. Thus, $\sum_i \psi_i(y_k) \to \sum_i z^i \geq 1$. In particular, some z^i must be greater than 0. Thus, the sequence y_k must eventually lie in U_i, so we may take $h_i(y_k)$ to be well defined past some point in the sequence. But then, since $\psi_i(y_k)h_i(y_k) \to \mathbf{z}^i$ and $\psi_i(y_k) \to z^i > 0$, then we must have

$$h_i(y_k) \to \frac{\mathbf{z}^i}{z^i}$$

as $k \to \infty$. But h_i is a homeomorphism, so this implies y_k converges to some point in $\overline{U}_i$, contradicting the fact that $y_k \to \infty$. Thus, we were wrong to assume $\Phi(x_n)$ had a convergent subsequence and we must have $\Phi(x_n) \to \infty$ in $\mathbb{R}^N$.