Topology Exercises

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Exercise 22

Problem: Let $M = \bigcup_{n \geq 1} K_n$ be such that K_n is compact and $K_n \subset int(K_{n+1})$. Define the refinement, $\mathcal{B}_n$ of this open cover as per the notes, and let $\mathcal{C}$ be defined similarly. Then $\mathcal{B}_{n+1}$ restricted to K_{n+1} has order $\leq m$. B_{n+1} consists of the following open sets:

1. Open sets of $\mathcal{B}_n$ intersecting K_{n-1} (and hence contained in K_n.)
2. Open sets of $\mathcal{C}$ which do not intersect K_n.
3. Open sets of the form

$$U' = \bigcup \{V \in \mathcal{C}; V \in \mathcal{C}_n \text{ and } f(V) = U\}$$

Solution: As per the hint, we note that $K_{n+1} = K_n \cup A_{n+1}$. We would like to show that $p \in K_{n+1}$ is in at most $m+1$ open sets of $\mathcal{B}_{n+1}$ which results in two cases.

Case 1: If $p \in A_{n+1}$, then any open set $B_p \in \mathcal{B}_{n+1}$ is either disjoint from K_n (and thus an element of $\mathcal{C}$ which does not intersect K_n) or is a union of open sets in $\mathcal{C}$. Since $\mathcal{C}$ has order $\leq m$ when restricted to A_{n+1}, then we are done.

Case 2: If $p \in K_n$, then any open set $B_p \in \mathcal{B}_{n+1}$ is either an open set in $\mathcal{B}_n$ which intersects K_{n-1} (and thus is contained in K_n by construction of $\mathcal{B}_n$), or is a union of open sets in $\mathcal{C}$ which are contained in an open set of $\mathcal{B}_n$. Now, $\mathcal{B}_n$ has order $\leq m$ when restricted to K_n.

In the case of an open set which intersects both K_n and A_{n+1}, we can regard the set as a union of its restrictions to both sets, which would satisfy the conditions of either case.

We conclude that B_{n+1} restricted to K_{n+1} has order $\leq m$