Topology Exercises

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Jordan-Brouwer Exercise 11

Utilizing the hint, we note that Exercise 10 implies that $\overline{D}_1$ is bounded. Thus taking its closure, we get a closed bounded subset of $\mathbb{R}^n$, which implies $\overline{D}_1$ compact. Furthermore, X is closed so $\mathbb{R}^n \setminus X$ is open. In particular, D_0 is open so $D_1 \cup X$ is closed. Now note that by (4), we can connect any point in D_1 to a point $x \in X$ with a curve. This defines a sequence in D_1 converging to the point x. Hence each point of x is a limit point of D_1 implying $\overline{D}_1 = D_1 \cup X$.

Now, let ψ be a local parametrization around a point $x \in X$ such that ψ maps an open ball $B(0) \subset \mathbb{R}^n$ diffeomorphically onto an open neighborhood, U of x such that $B \cap \mathbb{R}^{n-1}$ maps to $X \cap \psi(B)$. By (4), we can connect any point in $\mathbb{R}^n \setminus X$ to a point in U, and by (6) this points have the same winding number. Thus X splits U into two sets, one being a subset of D_1 and the other being a subset of D_0. Now we can consider the sets $B \cap H^n$ and $B \cap -H^n$. We see that $\psi(B \cap H^n)$ is connected so it is either mapped into $U \cap D_0$ or $U \cap D_1$ (similar for $\psi(B \cap -H^n)$). Finally, if $\psi(B \cap H^n)$ and $\psi(B \cap -H^n)$ are disjoint, for if not ψ would not be a bijection. Thus we see that either $\psi(B \cap H^n) \subset D_1$ and $\psi(B \cap -H^n) \subset D_0$ or vise versa. Thus we can restrict ψ to $\psi(B \cap H^n)$ which gives us a parameterization of $\overline{D}_1$, which concludes that $\overline{D}_1$ is a compact manifold with boundary, and we consider D_1 the "inside" and D_0 the "outside".