Math 562, Topology

Set 3

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Theorem: Suppose that X is the boundary of D, a compact manifold with boundary, and let $F : D \to \mathbb{R}^n$ be a smooth map extending F; that is, $\partial F = f$. Suppose that z is a regular value of F that does not belong to the image of f. Then $F^{-1}(z)$ is a finite set, and $W_2(f, z) = \left|F^{-1}(z)\right| \mod 2$.

Problem 2: Suppose that $F^{-1}(z) = \{y_1, \ldots, y_m\}$, and around each point y_k let B_k be a ball, or rather the image of a ball in $\mathbb{R}^n$ via some local parametrization of D. Demand that the balls are disjoint from one another and from the boundary $X = \partial D$. Let $f_k : \partial B_k \to \mathbb{R}^n$ be the restriction of F, and prove that

$$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_m, z) \mod 2.$$

Solution:

We may take the balls out of D to obtain a new manifold $\hat{D} = D \setminus (B_1 \cup \cdots \cup B_m)$. Notice that $F(\hat{D})$ doesn't contain z, hence $u : \partial\hat{D} \to S^{n-1}$ defined as

$$u(x) = \begin{cases} u_f(x) = \dfrac{f(x) - z}{|f(x) - z|} & x \in X \\ u_k(x) = \dfrac{f_k(x) - z}{|f_k(x) - z|} & x \in \partial B_k \end{cases}$$

can be extended to all of $\hat{D}$, and thus $\deg_2(u) = 0$. Moreover, X and all ∂B_k are disjoint from each other, therefore

$$0 = \deg_2(u) = \deg_2(u_f) + \deg_2(u_1) + \cdots + \deg_2(u_m) \mod 2.$$

After adding $\deg_2(u_f)$ to both sides we get

$$\deg_2(u_f) = \deg_2(u_1) + \cdots + \deg_2(u_m) \mod 2.$$

But this is the desired result by the definition of W_2. ∎