Analysis HW 6

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(3) Use the regularity of z to choose the balls B_i such that $W_2(f_i, z) = 1$, proving the theorem.

The hint provided states: "If f_i carries ∂B_i diffeomorphically onto a small sphere centered at z, then $u_i : \partial B \to S^{n-1}$ is bijective. But f is a local diffeomorphism at y_i, so we can choose the B_i that we want."

So, we know that z is a regular value of F, which implies that dF_z is a surjection. Not sure how to see that dF_z is also an injection, but if we could then the IFT would imply F is a diffeomorphism.

Now, for all i, we can find an open set $U \subset D$ with $y_i \in U$ and $V \subset \mathbb{R}^n$ with $z \in V$, such that $F : U \to V$ is a diffeomorphism. Now, because U, V are open, we can pick a clsoed $A_i \subset V$ such that F restricted to ∂B_i maps diffeomorphically to ∂A_i.

But, ∂A_i is a small sphere centered on z, so for $x_1, x_2 \in \partial B_i$, we have $f(x_1), f(x_2) \in \partial A_i$, so we can say

$$f(x_1) = z + rv_1$$
$$f(x_2) = z + rv_2$$

with $r =$ radius of A_i and $\in S^{n-1}$. Now we have $u : \partial B_i \to S^{n-1}$ and if

$$\begin{aligned} u(x_1) &= u(x_2) \\ \frac{f(x_1) - z}{|f(x_1) - z|} &= \frac{f(x_2) - z}{|f(x_2) - z|} \\ \implies \frac{rv_1}{|rv_1|} &= \frac{rv_2}{|rv_2|} \\ \implies v_1 &= v_2 \end{aligned}$$

So, we must have $x_1 = x_2$. Hence, u is injective. We can also see that u is surjective, for if $v \in S^{n-1}$, then $z + rv \in \partial A_i$, and then $u(f^{-1}(z + rv) = v$. Therefore, u is a bijetion, and we have that $u^{-1}(v)$ is just a single point. Thus, we have $[u^{-1}(v)] = I_2(u, \{v\}) = deg_2(u) = W_2(f_i, z) = 1$. Hence

$$W_2(f, z) = W_2(f_1, z) + \cdots W_2(f_i, z) = [F^{-1}(z)] \mod 2$$