MATH 562 - PROBLEM SET 3

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Exercise 4. Let $z \in \mathbb{R}^n - X$. Prove that if x is any point of X and U is any neighborhood of x in $\mathbb{R}^n$, then there exists a point of U that may be joined to z by a curve not intersecting X.

Proof. Let $z \in \mathbb{R}^n - X$. Let A be the set of all $x \in X$ for which the above holds, i.e. if $x \in A$ then for any open neighborhood U of x, there exists a point of U that may be joined to z by a curve not intersecting X. We will show that A is non-empty, closed, and open. By the connectedness of X, this will imply that $A = X$.

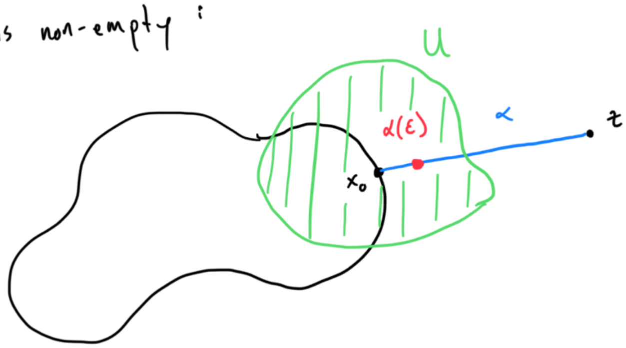
First, we show A is non-empty. Since X is assumed to be compact, there exists some $x_0 \in X$ such that $|z - x_0| = \inf_{x \in X} |z - x|$. Let $\alpha(t) = x_0 + t(z - x_0)$ for $t \in [0, 1]$ be the straight line path from x_0 to z. Then by the continuity of α, any neighborhood of x_0 contains $\alpha(\epsilon)$ for some $\epsilon > 0$, in which case α restricted to $[\epsilon, 1]$ is a curve from $\alpha(\epsilon) \in U$ to z. This curve does not intersect X, since this would contradict $|z - x_0| = \inf_{x \in X} |z - x|$ (explicitly, if $\alpha(t_0) \in X$ for some $t_0 > 0$, then $|z - \alpha(t_0)| = |(z - x_0)(1 - t_0)| < |z - x_0|$). Therefore, there exists some $x_0 \in A$.

Next, we show A is closed in X. Suppose x_n is a sequence of points in A converging to $x \in X$. Let U be an open neighborhood of x. Then there exists some $N \in \mathbb{N}$ such that $x_N \in U$. Find an open neighborhood V of x_N such that $V \subseteq U$. Since $x_N \in A$ by assumption, there exists a point in V and hence in U which may be joined to z by a curve not intersecting X. Thus $x \in A$, and A is closed.

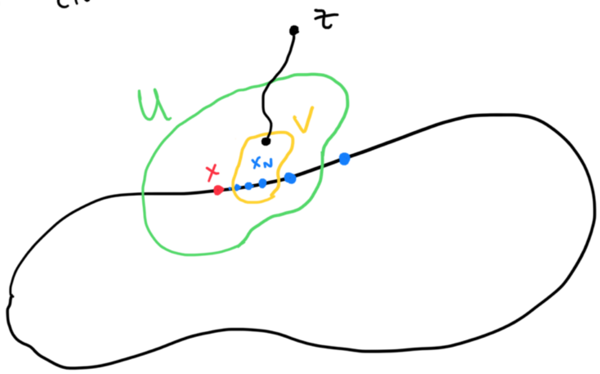
Finally, we show A is open. Let $x_0 \in A$. By the local immersion theorem, there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0 and a coordinate chart $\phi : U \to \mathbb{D}^n$ ($\mathbb{D}^n$ is the unit n-disk) such that $\phi(U \cap X) = \mathbb{D}^n \cap (\mathbb{R}^{n-1} \times \{0\})$. Since $x_0 \in A$, there exists a point, say $u \in U$, that may be joined to z by a curve, say α, not intersecting X. Note $\phi(u)$ is either in the upper half of the disk $\mathbb{D}^n \cap (\mathbb{R}^{n-1} \times (0, \infty))$ or the lower half $\mathbb{D}^n \cap (\mathbb{R}^{n-1} \times (-\infty, 0))$. In either case, if $x \in U \cap X$ and V is a neighborhood of x in $\mathbb{R}^n$ (which we may assume to be contained in U by shrinking if necessary), then $\phi(V)$ intersects both the upper half and lower half of $\mathbb{D}^n$, hence there exists $v \in V$ such that $\phi(v)$ may be joined to $\phi(u)$ by a curve β in $\mathbb{D}^n$ not intersecting $\mathbb{R}^{n-1} \times \{0\}$. Then $\phi^{-1} \circ \beta$ and α together define a path from $v \in V$ to z not intersecting X. Hence $x \in A$. Thus $U \cap X \subset A$, and so A is open in X. □

Date: March 16, 2021.

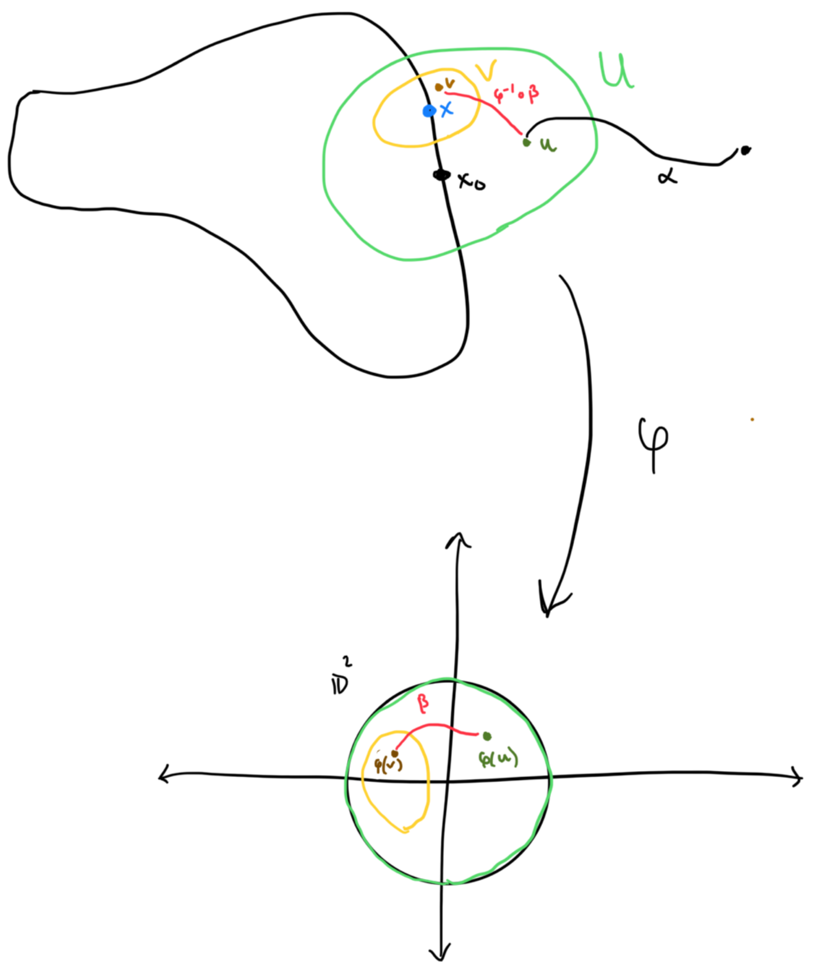
A is non-empty:



A is closed:



A is open: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$\varphi^{-1}\circ\beta$
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