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Math 562

Problem Set 3

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Exercise 6

Show that if z_0 and z_1 belong to the same connected component of $\mathbb{R}^n \setminus X$, then $W_2(X, z_0) = W_2(X, z_1)$.

X is compact, connected hypersurface in $\mathbb{R}^n$.
$W_2(X, z)$ is the winding number of the inclusion map of X around z.

Proof :

X compact $\implies \mathbb{R}^n \setminus X$ is open.
$\implies$ Connected components of $\mathbb{R}^n \setminus X$ are path connected.

Since z_0 and z_1 are in the same connected component, we can join z_0 and z_1 by a curve $z_t : [0,1] \to \mathbb{R}^n \setminus X, \qquad t \in [0,1]$.

Define the homotopy, $u_t(x) := \dfrac{x - z_t}{|x - z_t|}$

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$$u_0(x) := \frac{x - z_0}{|x - z_0|}$$
$$u_1(x) := \frac{x - z_1}{|x - z_1|}$$

Now, $W_2(X, z_0) = \deg_2(u_0)$, and
$W_2(X, z_1) = \deg_2(u_1)$.

But, by the Theorem in G-P (p-81), **Homotopic maps have the same mod 2 degree**.

Therefore, $W_2(X, z_0) = W_2(X, z_1)$.