Problem set 3, Problem 7

Given a point $z \in R^n - X$ (X is the boundary of a compact manifold with boundary on R^n) and $v \in S^{n-1}$, consider the ray $r = \{z + tv : t \geq 0\}$. Check that the ray r is transversal to X if and only if v is a regular balue of direction map $u : X \to S^{n-1}$. In particular, almost every ray from z intersects X transversally.

Proof. By translating the R^m so that the origin goes to z, we can assume, without loss of generality, that $z = 0$. First, note that we can write

$$u(x) = \frac{x}{\|x\|}.$$

This is well defined as $x \neq 0$, because $0 = z \in R^n - X$ implies $0 \notin X$. Then, using product rule, we have

$$du(x)v = d\left(\frac{1}{\|x\|}\right)(x \cdot v) + d(x)\frac{v}{\|x\|} = d\left(\frac{1}{\|x\|}\right)(x \cdot v) + \frac{v}{\|x\|}.$$

Now,

$$d\left(\frac{1}{\|x\|}\right) = d\left(\frac{1}{\sqrt{\sum_{i=1}^m x_i^2}}\right) = \left(-\frac{x_1}{(\sum_{i=1}^m x_i^2)^{3/2}}, \ldots, -\frac{x_m}{(\sum_{i=1}^m x_i^2)^{3/2}}\right) = -\frac{x}{\|x\|^3}.$$

Therefore,

$$du(x)v = \frac{1}{\|x\|}\left(v - \frac{x \cdot v}{\|x\|^2}x\right) = \frac{1}{\|x\|}\left(v - \left(\frac{x}{\|x\|} \cdot v\right)\frac{x}{\|x\|}\right).$$

Observe that $\left(v - \left(\frac{x}{\|x\|} \cdot v\right)\frac{x}{\|x\|}\right)$ is the projection of v onto the orthogonal space of the vector $\frac{x}{\|x\|} = u(x)$. Let $P_{u(x)}^{\perp}$ denote the linear transformation of projection onto orthogonal space of vector $u(x)$. Using this notation, we obtain a simple expression for $du(x)$:

$$du(x) = \frac{1}{\|x\|}P_{u(x)}^{\perp}.$$

We are ready to solve the problem:

($\Longrightarrow$) Suppose v is transversal to X. Then either r does not intersect X, or, since $v = dr$, we have $\text{span}(v) + T_xX = R^n$ for each $x \in r \cap X$. In particular, If r does not intersect X, since $z = 0$, this implies $v \notin u(X)$. So, $u^{-1}(v) = \emptyset$. So v is vacuously a regular value of u.

If r does intersect X, let x be a point of such intersection. Then, since $z = 0$, we can deduce that $x \in u^{-1}(v)$.' Since X denotes a boundary of a compact manifold with boundary, $T_xX \neq R^n$ as $\dim(T_xX) \leq n$. Thus, $\text{span}(v) + T_xX = R^n$ implies $v \notin T_xX$. As T_xX is $n-1$ dimensional, there is only one choice of T_xX:

$$T_xX = \text{Im}(P_v^{\perp}).$$

Now, we can observe geometrically that $T_v S^{n-1} = \text{Im}(P_v^\perp)$. Let $y \in T_v S^{n-1}$. Then $\|x\| y \in \text{Im}(P_v^\perp)$, and

$$du(x)(\|x\|y) = \frac{\|x\|}{\|x\|}\left(P_v^\perp y\right) = y,$$

which demonstrates surjectivity of $du(x) : T_x X \to T_v S^{n-1}$. Since $x \in u^{-1}(v)$ was arbitrary, v is a regular point of u.

($\Longleftarrow$) Let v be a regular point of u. Then, since $z = 0$, either $v \notin u(X)$, or $du(x)$ is surjective for $x \in u^{-1}(v) = r \cap u(X)$. Again, because $z = 0$, $v \notin u(X)$ implies $r \cap X = \emptyset$, which implies transversality.

To consider the other case, fix $x \in r \cap u(X)$. Now, since the codomain of $du(x)$ is $T_v S^{n-1}$, which is $n-1$ dimensional; and the domain of $du(x)$, which is at most $T_x X$, is $n-1$ dimensional; and $du(x)$ is surjective, we must have $du(x)$ be an isomorphism. This, in turn, implies $T_x X$ is precisely $n-1$ dimensional. Note that $v \notin T_x X$ as,

$$du(x)v = \|x\| P_v^\perp v = 0,$$

contradicts the fact that $du(x)$ is isomorphic from $T_x X$ to $T_v S^{n-1}$. Since $T_x X \subset R^n$ is a $n-1$ dimentional linear space with $v \notin T_x X$, there is only 1 choice for $T_x X$:

$$T_x X = \text{Im}(P_v^\perp).$$

Thus, for each $x \in r \cap u(X)$,

$$T_x r + T_x X = \text{span}(v) + \text{Im}(P_v^\perp) = R^n,$$

which demonstrates transversality. □