Topology Exercises 3

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Problem 9. Prove that $\mathbb{R}^n \setminus X$ has precisely two components,

$$D_0 = \{z : W_2(X, z) = 0\} \quad \text{and} \quad D_1 = \{z : W_2(X, z) = 1\}.$$

Proof. First, note that D_0 and D_1 are disjoint and $D_0 \cup D_1 = \mathbb{R}^n \setminus X$.

Due to exercise 8, we know that both sets above are nonempty—because if we choose some $z_0 \in \mathbb{R} \setminus X$, then z_0 must lie in either D_0 and D_1. Problem 7 showed that almost every ray from z_0 intersects X transversally (and hence in a finite set), so simply choose a ray r from z which intersects X transversally and nontrivially (i.e., so $r \cap X \neq \emptyset$) and then choose a point $z_1 \in r \setminus X$ with only a single intersection point of X and r between it and z_0. Exercise 8 shows us that if $z_0 \in D_0$ then $z_1 \in D_1$, and vice-versa.

Now, if z is in one of these sets then there is a connected neighborhood U of z which does not intersect X. But then, by problem six, any point of U has the same mod 2 winding number as z, hence U is a subset of the set we picked z from.

The previous steps show that D_0 and D_1 form a separation of $\mathbb{R}^n \setminus X$, so by problem 5 they must also be connected, for otherwise $\mathbb{R}^n \setminus X$ would have more than two connected components. □