Topology HW 1

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3 Show that the following two conditions define a topology on TM making it a topological manifold. Describe a local basis of neighborhoods at a point (p, X_p) of TM.

(a) π is a continuous map: $\pi^{-1}(U)$ is open in TM if $U \subset M$ is an open set

(b) Let (U, h) be a local chart for M, $h(p) = 0$. Define $\tilde{h} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ by

$$\tilde{h}(p, X_p) = (h(p), v)$$

where $X_p = \sum_i v^i \partial_{i|p}$, the components of X_p in the basis $\partial_{i|p}$ of T_pM associated with the chart h. We require $\tilde{h}$ to be a homeomorphism.

We must show a few things: locally Euclidean, Hausdorff, and the existence of a 2nd countable basis. In order to see that TM is locally Euclidean, look at $\tilde{h}$. It takes $\pi^{-1}(U)$ to $U \times \mathbb{R}^m$. But h is a coordinate map, so $U \subset \mathbb{R}^m$, so we have that $\pi^{-1}(U)$ is homeomorphic to an open set in $\mathbb{R}^{2m}$. Thus, we have that the tangent bundle is locally Euclidean.

We now show that TM is Hausdorff. So, let $(p, X_p) \neq (q, X_q)$. We have two cases: if $p \neq q$ or $p = q$. In the first case, we use the fact that M is Hausdorff: we know that there exist disjoint open neighborhoods of p, q, call them U_p, U_q respectively. Since π is continuous, we know that $\pi^{-1}(U_p)$ and $\pi^{-1}(U_q)$ are disjoint open neighborhoods of (p, X_p) and (q, X_q).
Now for the second case, we have $p = q$ and $X_p \neq X_q$. Let U be an open coordinate neighborhood of $p = q$. By the condition (b) above, we know that $\tilde{h} : \pi^{-1}(U) \to h(U) \times \mathbb{R}^m$ is homeomorhpism, but since $(p, X_p) \neq (q, X_q)$, we must have that $\tilde{h}(p, X_p) \neq \tilde{h}(q, X_q)$. So we have $\tilde{h}(p, X_p) = (0, v)$ and $\tilde{h}(q, X_q) = (0, w)$. Now, let $A_v, A_w \subset \mathbb{R}^m$ be disjoint. Finally, we then have that $\tilde{h}^{-1}(h(U) \times A_v)$ and $\tilde{h}^{-1}(h(U) \times A_w)$ are disjoint open neighborhoods of (p, X_p) and (q, X_q) respectively.

Now we show that TM is second-countable, so we need a countable basis for the topology on TM. First let $\{U_n\}_{n \geq 1}$ be a countable basis for the topology on M, consisting of coordinate neighborhoods. Similarly, let $\{A_i\}_{i \geq 1}$ be a countable basis for the standard topology on $\mathbb{R}^m$. Now we define the function:

$$\tilde{h}_n : \pi^{-1}(U_n) \to h_n(U_n) \times \mathbb{R}^m$$

where $h_n : U_n \to \mathbb{R}^m$ are coordinate charts. Now we finally define the collection of sets $\{W_{ni}\} := \tilde{h}^{-1}(h_n(U_n) \times A_i)$. These sets W_{ni} are a countable basis for the topology on TM.