MATH 562 - PROBLEM SET 1

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Exercise 4. Let $\pi : E \to M$ be a locally trivial vector bundle with fiber $\mathbb{V}$ and $\dim \mathbb{V} = n$. Show that $\pi : E \to M$ is trivial if and only if there exists differentiable sections $X_1, X_2, \cdots, X_n : M \to E$ such that the set $\{X_i(p)\}$ is a basis of $\pi^{-1}(p) = E_p$, for all $p \in M$.

Proof. Suppose $\pi : E \to M$ is trivial and let $\Phi : E \to M \times \mathbb{V}$ be the global trivialization. Define differentiable maps $X_i : M \to M \times \mathbb{V}$ by $X_i(p) = (p, \mathbf{e}_i)$ where $\mathbf{e}_i$ is the i-th basis element in some fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ of $\mathbb{V}$. Then $\Phi^{-1} \circ X_i$ are our desired differentiable sections.

So now suppose that there exist differentiable sections $X_1, X_2, \cdots, X_n : M \to E$ such that the set $\{X_i(p)\}$ is a basis of $\pi^{-1}(p) = E_p$, for all $p \in M$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be a basis of $\mathbb{V}$. Define $\Phi : M \times \mathbb{V} \to E$ by

$$\Phi\Big(p, \sum_{i=1}^{n} \lambda_i \mathbf{e}_i\Big) = \sum_{i=1}^{n} \lambda_i X_i(p)$$

where the right hand side is to be understood as a sum in the vector space $E_p \cong \mathbb{V}$. Since $\{X_i(p)\}$ is a basis of E_p for all $p \in M$, Φ is bijective. To show that Φ is a diffeomorphism, we show that it is a local diffeomorphism.

Let $p_0 \in M$. Then by the definition of $\pi : E \to M$, we may find an open neighborhood U of p_0 such that there exists a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{V}$. If $\phi \circ \Phi|_{U \times \mathbb{V}}$ is a diffeomorphism, then $\Phi|_{U \times \mathbb{V}}$ is a diffeomorphism, which would show that Φ is a local diffeomorphism since $p_0 \in M$ was arbitrary.

For each section $X_i : M \to E$, the map $\phi \circ X_i|_U : U \to U \times \mathbb{V}$ is smooth, hence there exist smooth functions $X_i^j : U \to F$, where F is the field of scalars for $\mathbb{V}$, such that

$$\phi \circ X_i|_U(p) = \Big(p, \sum_{j=1}^{n} X_i^j(p) \mathbf{e}_j\Big).$$

The composition $\phi \circ \Phi|_{U \times \mathbb{V}}$ is then defined by

$$\Big(p, \sum_{i=1}^{n} \lambda_i \mathbf{e}_i\Big) \mapsto \Big(p, \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i X_i^j(p) \mathbf{e}_j\Big)$$

and is smooth since it is a linear combination of smooth functions. Moreover, for each $p \in U$, since $\{X_i(p)\}$ is a basis of E_p, $\phi \circ \Phi$ is invertible at p. Hence the inverse of $\phi \circ \Phi|_{U \times \mathbb{V}}$ exists and can be seen to be smooth (for example since matrix inversion is a smooth map from $GL(\mathbb{V})$ to itself). Thus $\phi \circ \Phi|_{U \times \mathbb{V}}$ is a diffeomorphism. $\square$

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