Problem Set 1

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Exercise 5.

(i) If $f, g \in C(S^n, S^n)$ and $f(x) + g(x) \neq 0$ for all $x \in S^n \subseteq \mathbb{R}^{n+1}$, then f is homotopic to g.

(ii) In particular, if $f \in C(S^2, S^2)$ has no fixed points, f is homotopic to the antipodal map, α.

(iv) Suppose V were a nonvanishing vector field on S^2. Consider the map

$$f(x) = \frac{x + V(x)}{\|x + V(x)\|}.$$

f is homotopic to the identity but has no fixed points.

(i') If f, g satisfy $\|f(x) + g(x)\| < 2$ for all $x \in S^n$, then f is homotopic to g.

Proof. (i) Note that since $f(x) \neq -g(x)$, we have that the path $k(t) = tf + (1-t)g$ is continuous in $\mathbb{R}^{n+1}$, and avoids the origin, so we may project it onto the sphere S^n by taking

$$h(t) = \frac{tf + (1-t)g}{\|tf + (1-t)g\|}.$$

(ii) Using (i) and taking $g = \alpha$, we have that $f(x) + \alpha(x) = f(x) - x \neq 0$ if $f(x) \neq x$ (that is, f has no fixed points). Then f and α are homotopic.

(iv) Utilizing the hint given, take

$$k(t) = \frac{x + tV(x)}{\|x + tV(x)\|}.$$

This is a path from the identity f, showing that they are homotopic. On the other hand, as V is nonvanishing, f has no fixed points, so f is homotopic to the antipodal map. But this gives a contradiction, as the identity and the antipodal map are not homotopic.

(i') Since we are considering the unit sphere, we have for any $x \in S^n$, $\|x - (-x)\| = 2\|x\| = 2$, so if $\|f(x) + g(x)\| < 2$, we must have that $f(x) \neq -g(x)$ for each $x \in S^n$. Now using (i), we have that f is homotopic to g.

$\square$