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Math 562
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# Problem Set 1

## Exercise 6

**(i) Generalizing Example 1, show that any odd-dimensional sphere admits a non-vanishing vector field.**

***Example 1*** : $TS^1$ is trivial: $V(x, y) = (-y, x)$ is a nonvanishing tangent vector field on $S^1$.

**Proof :**
We can write an odd-dimensional sphere as $S^{2n-1}$.
Now, let us consider $S^{2n-1}$ as the unit sphere in $\mathbb{C}^n$, complex $n-$dimensional space:

$$S^{2n-1} = \{(z_1, \cdots, z_n); \sum_j |z_j|^2 = 1, z_j = u_j + iv_j.\}$$

Let $V(z_1, \cdots, z_n) = (z_2, -z_1, z_4, -z_3, \cdots, z_n, -z_{n-1})$

To see that $V(z_1, \cdots, z_n)$ is tangent to $S^{2n-1}$ everywhere:

$$\begin{aligned} &\langle (z_1, \cdots, z_n) \cdot (z_2, -z_1, z_4, -z_3, \cdots, z_n, -z_{n-1}) \rangle \\ &\qquad = \text{Re}(-z_1\overline{z}_2 + z_2\overline{z}_1 - z_3\overline{z}_4 + z_4\overline{z}_3 - \cdots + z_{n-1}\overline{z}_n - z_n\overline{z}_{n-1}) \\ &\qquad = 0 \end{aligned}$$

Also $V(z_1, \cdots, z_n)$ vanishes only at $0 \in C^n$.
$(z_2, -z_1, z_4, -z_3, \cdots, z_n, -z_{n-1}) = 0$ implies $z_j = 0, \forall j$

**(ii) Show that the 3-sphere is *parallelizable* (i.e. the tangent bundle $TS^3$ is trivial).**

Similar to $S^1$, where we can identify $(x, y) \in S^1$ with $x+iy \in \mathbb{C}$, we can identify $(x_1, x_2, x_3, x_4) \in S^3$ with $x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$.

Now, notice that for $S^1$ we multiplied $x + iy$ by $i$ to find $i(x + iy) = -y + ix \in \mathbb{C}$ or $(-y, x) \in \mathbb{R}^2$ a non-vanishing vector field.

Similarly, for $S^3$, we can find 3 non-vanishing vector fields multiplying $((x_1, x_2, x_3, x_4) \in S^3$ by $i, j,$ and $k$.

$$v_1 = i(x_1 + ix_2 + jx_3 + kx_4) = ix_1 - x_2 + kx_3 - jx_4 = -x_2 + ix_1 - jx_4 + kx_3$$
$$v_2 = j(x_1 + ix_2 + jx_3 + kx_4) = jx_1 - kx_2 - x_3 + ix_4 = -x_3 + ix_4 + jx_1 - kx_2$$
$$v_3 = k(x_1 + ix_2 + jx_3 + kx_4) = kx_1 + jx_2 - ix_3 - x_4 = -x_4 - ix_3 + jx_2 + kx_1$$

So, $v_1(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$,
$v_2(x_1, x_2, x_3, x_4) = (-x_3, x_4, x_1, -x_2)$, and
$v_3(x_1, x_2, x_3, x_4) = (-x_4, -x_3, x_2, x_1)$ are three non-vanishing tangent vector fields on $S^3$.

Independence of $v_1, v_1, and v_3$ follows from the fact that it is a list of orthonormal vectors.