Problem set 1, Problem 7

If $f : U \to R^m$, $U \subset R^m$ open, is a C^1 immersion, show that $V = f(U)$ is open in R^m, and that if f is injective, then f is a C^1 diffeomorphism from U onto V.

Proof. We will make extensive use of the Implicit Function Theorem:

Theorem 1. *Let $U \subset R^m$ be open, $f : U \to R^m$ be a C^r map ($r \geq 1$). Suppose $df(x) \in \mathcal{L}(R^m)$ is an isomorphism for some $x \in U$. Then, there exists open neighborhoods U of x, V os $f(x)$ in R^m such that*

(i) f is a homeomorphism from U onto V.

(ii) $f^{-1} : V \to U$ is of class C^r, with $df^{-1}(f(x)) = (df(x))^{-1} \in \mathcal{L}(R^m)$.

Now, we show that V is open. Let $y \in V$, and let $x \in f^{-1}(\{y\})$. Note that since $U \subset R^m$ open, we know that $df(x)$ is an $m \times m$ matrix. Since f is an immersion, $df(x)$ must have rank m. But, since $df(x) \in \mathcal{L}(R^m)$, pocessing rank m implies that $df(x)$ is invertible. Thus, the linear map corresponding to $df(x)$ is an isomorphism. By Implicit Function Theorem (i), there exists neighborhoods W_x of x and Z_y of y such that f is a homeomorphism from W_x to Z_y. Homeomorphism implies bijectivity, and $W_x \subset U$, so $f(W_x) = Z_y \subset f(U) = V$. Thus, Z_y is an open neighborhood of y contained in V. Then $V = \bigcup_{y \in V} Z_y$ so that V is open.

Now, if we suppose that f is injective, we immediately deduce that f is bijective onto V. Then, by Implicit function Theorem (ii), for each $y \in V$, f^{-1} is C^1 differentiable at y. y was arbitrary, so $f^{-1} : V \to U$ must be a C^1 map. This, combined with the fact that $f : U \to V$ is a bijective C^1 map, implies, by defition of diffeomorphism, that f is a diffeomorphism onto V. $\square$