TOPOLOGY EXERCISES

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Exercise 9. First, suppose that $M \subset \mathbb{R}^n$ is a submanifold of dimension m of $\mathbb{R}^n$. Now, for any $p \in M$, there exists a neighborhood U of p in $\mathbb{R}^n$ and a C^r diffeomorphism $\phi : U \to \mathbb{R}^n$ so $\phi(M \cap U) \subset \mathbb{R}^m \times \{0\}^{n-m}$. But then, we may as well consider the map $\tilde{\phi} : M \cap U \to \mathbb{R}^m$ given by the natural projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$ composed with ϕ. This is a diffeomorphism, so $\tilde{\phi}^{-1}$ is an injective immersion of the image $\tilde{\phi}(M \cap U) \subset \mathbb{R}^m$ into an open neighborhood U of $\mathbb{R}^n$ which is also a homeomorphism. We know that $\tilde{\phi}$ is an immersion, because its inverse is the composition of a projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ which is equal to the identity on $\phi(M \cap U)$, and a diffeomorphism ϕ. Then, $d\tilde{\phi}^{-1} = d\pi \circ d(\phi)$ is an invertible linear map from T_pM to $T_{\pi\circ\phi(p)}\mathbb{R}^m$. Thus, M is also a C^r surface in $\mathbb{R}^n$.

Now, conversely, let M be a C^r surface of dimension m in $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$. By the definition of a surface, we have an immersion $\psi : U \to M$ for some open $U \subset \mathbb{R}^m$ and an open $Z \in \mathbb{R}^n$, for which $p \in \psi(U)$, and which is a homeomorphism onto its image. By the local form of immersions, we may appropriately choose the open neighborhood Z of p in $\mathbb{R}^n$ and a C^r diffeomorphism $\varphi : Z \to \mathbb{R}^n$ so the composition $\Phi = \psi \circ \varphi : U \to \mathbb{R}^n$ satisfies $\Phi(x) = (x, 0)$. Thus, the restriction $\psi|_{M\cap Z} : M \cap Z \to \mathbb{R}^m$ is a coordinate chart of M compatible with the smooth structure on M, being a diffeomorphism from the surface to an open subset of $\mathbb{R}^m$. Since p was arbitrary, M is a submanifold of $\mathbb{R}^n$.