GEODESICS OF A LORENTZIAN MANIFOLD.

A C^1 immersed curve $\gamma(\tau)$ in a Lorentzian manifold (M, g) (g with signature $(-, + \ldots, +)$) is *timelike* if $g(\dot\gamma, \dot\gamma) < 0$; *null* if $g(\dot\gamma, \dot\gamma) = 0$; *spacelike* if $g(\dot\gamma, \dot\gamma) > 0$. In particular this terminology applies to affinely parametrized geodesics; in this case, the sign of $g(\dot\gamma, \dot\gamma)$ is constant on γ.

Causal structure. Two points $p, q \in M$ are *chronologically related* if there exists a C^1 timelike curve in M from p to q. If the timelike curve is future-directed from p to q, q is in the *chronological future* of p. Notation:

$$p << q, \text{ or } q \in I^+(p) \text{ or } p \in I^-(q).$$

$I^+(p)$ and $I^-(p)$ are always open sets. q is in the *causal future* of p if either $p << q$ or there exists a null geodesic from p to q. Notation:

$$p < q \text{ or } q \in J^+(p) \text{ or } p \in J^-(q).$$

$J^+(p)$ and $J^-(p)$ are in general neither open nor closed.

If $q \in I^+(p)$ a timelike geodesic from p to q- if it exists- will *maximize* the functional:

$$\int_c \sqrt{-g(\dot\gamma, \dot\gamma)}\, d\tau$$

among C^1 timelike arcs c connecting the points p_1 and q_1 of γ, assuming q_1 is in a normal neighborhood of p_1. (If the affine parameter is chosen so that $g(\dot\gamma, \dot\gamma) = -1$, this functional is called *proper time* along γ.)

Definition. A Lorentzian manifold (M, g) is *timelike geodesically complete* the maximal domain of definition of any affinely parametrized timelike geodesic is $\mathbb{R}$. (An affine parametrization is defined by the property $\nabla_{\dot\gamma}\dot\gamma \equiv 0$). *Null geodesically complete* and *spacelike geodesically complete* are defined similarly, and *complete* means all three hold.

In contrast with the Riemannian Hopf-Rinow theorem, for general complete Lorentzian manifolds M it is possible to find points $p, q \in M$ such that $p << q$ but there is no timelike geodesic from p to q. In this note we give the details of the classical example of this phenomenon(see the figure in [1, p.7], or on the cover of [2]).

Remark: in terms of the physical interpretation this is not so surprising: $p << q$ means the events p and q could potentially be on the worldline of a massive object (with p preceding q), while a timelike geodesic arc from p to q represents the worldline of a massive object in 'free fall', that is, subject only to the gravitational field. You wouldn't expect all events in your life history to be potentially connected just by 'free fall', right?

The example is the Lorentzian metric defined in the open set $U \subset \mathbb{R}^2$:

$$ds^2 = \frac{dx^2 - dt^2}{\cos^2 x}, \quad U = \{(x,t) \in \mathbb{R}^2; |x| < \pi/2\}.$$

Since the metric is conformal to the 'Minkowski metric in U' $ds_U^2 = -dt^2 + dx^2$, the light cones at each point are the same for both metrics, and hence the 'causal structure' (e.g. the sets $I^{\pm}(p)$ and $J^{\pm}(p)$) is also the same.[1]

The differential equation for affinely parametrized geodesics $\gamma(\tau) = (x(\tau), t(\tau))$ is:

$$\begin{cases} \ddot{x} = -(\tan x)[(\dot{x})^2 + (\dot{t})^2] \\ \ddot{t} = -2(\tan x)\dot{x}\dot{t} \end{cases}$$

Exercise 1: derive the equations. In addition, you should verify that the quantity $[(\dot{t})^2 - (\dot{x})^2]/\cos^2 x$ is conserved along solutions.

For timelike (resp. null, spacelike) geodesics, the usual normalizations (by proper time and arc length in the first and third cases, resp.) are:

$$\begin{aligned} &(\dot{t})^2 - (\dot{x})^2 = \cos^2 x && \text{(timelike);} \\ &(\dot{t})^2 - (\dot{x})^2 = 0 && \text{(null);} \\ &(\dot{x})^2 - (\dot{t})^2 = \cos^2 x && \text{(spacelike).} \end{aligned}$$

The geodesics in this example admit a second conserved quantity.
Exercise 2: Show that, along each affinely parametrized geodesic:

$$\frac{\dot{t}}{\cos^2 x} = C,$$

where C is a constant (depending on the geodesic and choice of affine parameter.)

Exercise 3: Show that the 'slices' $t = const.$ are totally geodesic, and derive the differential equation for geodesics on a slice.

1.Completeness. (a) We first consider *null* geodesics. Assume $\gamma(\tau)$ is a future-directed null geodesic from (x_0, t_0), say with $\dot{x}(0) = \dot{t}(0) > 0$, defined

[1] However, note that unlike $ds^2 = -dt^2 + dx^2$ in $\mathbb{R}^{1+1}$, ds_U^2 is not 'globally hyperbolic' (for instance, $J^+((0,-2)) \cap J^-((0,2))$ is not compact). ds_U^2 is only 'causally simple' ($J^+(q)$ and $J^-(q)$ are closed in U for each $q \in U$.)

for $\tau \in I$ (maximal open interval). Then $\dot{x}(\tau) = \dot{t}(\tau)$ for each $\tau \in I$, hence $x(\tau) = t(\tau) + x_0 - t_0$, and from Exercise 2 $x(\tau)$ is the solution of the (easily integrated) first-order initial-value problem:

$$\dot{x} = A\cos^2 x, \quad x(0) = x_0, \quad A = \frac{\dot{x}(0)}{\cos^2 x_0}.$$

This shows $|\dot{x}|$ is 'a priori-bounded' (depending only on initial conditions), and from the equation:

$$\ddot{x} = -2(\tan x)(\dot{x})^2 = -2A^2 \sin x \cos x,$$

we see the same is true for $|\ddot{x}|$. Thus $I = \mathbb{R}$, and in particular $\gamma(\tau)$ does not reach the boundary $x = \pm\pi/2$ for finite τ.

(b) Turning to *timelike* geodesics, let $\gamma(\tau) = (x(\tau), t(\tau))$ be a future-directed timelike geodesic parametrized by proper time $\tau \in I$ (maximal interval). We have, for some constant $C > 0$ (depending on initial conditions):

$$\dot{t} = C\cos^2 x, \quad (\dot{x})^2 = (\dot{t})^2 - \cos^2 x = (C^2\cos^2 x - 1)\cos^2 x,$$

so clearly $|\dot{x}|$ and $|\dot{t}|$ are a-priori bounded, depending only on initial conditions. In addition, $C^2\cos^2 x \geq 1$, or $\cos x \geq 1/C$ for $\tau \in I$ (with equality only where $\dot{x} = 0$), so $x(\tau)$ is bounded away from the boundary $\pm\pi/2$, and $\tan x$, and therefore $\ddot{x}$ and $\ddot{t}$, are all a-priori bounded; so again $I = \mathbb{R}$.

(c) *Spacelike* geodesics. Showing spacelike completeness is *Exercise 4.*

2. Integrability. Consider a future-directed timelike geodesic arc (parametrized by proper time) $\gamma(\tau) = (x(\tau), t(\tau))$, on which $\dot{x} \neq 0$ (say, $\dot{x} > 0$), starting at $\gamma(\bar{\tau}) = (\bar{x}, \bar{t})$. On such an arc t is an increasing function of f, with derivative:

$$\frac{dt}{dx} = \frac{\dot{t}}{\dot{x}} = \frac{C\cos x}{\sqrt{C^2\cos^2 x - 1}},$$

where $C = \dot{t}(\bar{\tau})/\cos^2\bar{x} > 1/\cos x$, as seen in (b) above. This can be integrated appealing to the change of variables $y = C\sin x$:

$$t(x) - \bar{t}(\bar{x}) = \int_{\bar{x}}^{x} \frac{C\cos x' dx'}{\sqrt{C^2\cos^2 x' - 1}} = \int_{\bar{y}}^{y} \frac{dy'}{\sqrt{C^2 - 1 - (y')^2}} = \arcsin\left(\frac{y}{\sqrt{C^2-1}}\right) - \arcsin\left(\frac{\bar{y}}{\sqrt{C^2-1}}\right).$$

Note that $\dot{x} = 0$ iff $\cos x = 1/C$. Extending this for negative time until the first point on γ where $\dot{x} = 0$, we may assume $\cos\bar{x} = 1/C$, so $\sin\bar{x} = \pm(1/C)\sqrt{C^2-1}$ and $\bar{y} = \pm\sqrt{C^2-1}$, and therefore:

$$\arcsin\left(\frac{\bar{y}}{\sqrt{C^2-1}}\right) = \pm\frac{\pi}{2}, \quad (+ \text{ if } \bar{x} > 0, - \text{ if } \bar{x} < 0).$$

(Note $\bar{x} \neq 0$ necessarily, unless γ is contained in the vertical geodesic $\{x = 0\}$, the only geodesic of the form $\{x = const.\}$.) This is easily seen to imply:

$$\cos(t - \bar{t}) = \pm \frac{C}{\sqrt{C^2 - 1}} \sin x, \quad \text{where } \dot{x}(\bar{\tau}) = 0;$$

we take (+) if $\bar{x} > 0$ (so x decreases with τ on the arc γ), (-) if $\bar{x} < 0$ (x increases with τ).

This may be thought of as an 'equation' for the geodesic arc, valid as long as $\dot{x} > 0$. It implies, in particular, that if $\bar{x} < 0$ at the lower endpoint of the arc, then at some point on the arc we have $x = 0$ (namely, when $t = \bar{t} + \pi/2$.)

3. Symmetry. The geodesic flow on this manifold has a reflection symmetry property. Consider a complete future-directed affinely parametrized timelike geodesic $\gamma(\tau) = (x(\tau), t(\tau))$, $\tau \in \mathbb{R}$. Assume $\dot{x}(\tau_0) = 0$, and consider the timelike curve:

$$\gamma_1(\tau) = (x(\tau_0 - \tau), 2t(\tau_0) - t(\tau_0 - \tau)), \quad \tau \in \mathbb{R}.$$

Exercise 5: $\gamma_1(\tau)$ is a geodesic satisfying $\gamma_1(0) = \gamma(\tau_0), \dot{\gamma_1}(0) = \dot{\gamma}(\tau_0)$.

By uniqueness, if follows that $\gamma_1(\tau) = \gamma(\tau + \tau_0)$ for all $\tau \in \mathbb{R}$, that is: $x(\tau_0 + \tau) = x(\tau_0 - \tau)$ and $t(\tau_0 + \tau) + t(\tau_0 - \tau) = 2t(\tau_0)$. This means γ is symmetric under reflection on the line $\{t = t(\tau_0)\}$:

$$(\bar{x}, \bar{t}) \in \gamma(\mathbb{R}) \Rightarrow (\bar{x}, 2t(\tau_0) - \bar{t}) \in \gamma(\mathbb{R}) \quad (\text{ assuming } \dot{x}(\tau_0) = 0).$$

It is not hard to see that a parameter value τ_0 with $\dot{x}(\tau_0) = 0$ must exist. Say, for instance, $x(\tau_1) > 0$ and $\dot{x}(\tau_1) > 0$ for some τ_1. Then $\tan x(\tau) > \tan x(\tau_1) > 0$ for $\tau > \tau_1$, as long as $\dot{x}(\tau) > 0$. Then:

$$(\dot{x})^2 + (\dot{t})^2 = 2(\dot{x})^2 + \cos^2 x \geq \cos^2 x \geq \frac{\cos^4 x(\tau_1)}{\dot{t}(\tau_1)^2},$$

(see 1(b)) so $\ddot{x} \leq -(\sin x(\tau_1) \cos^3 x(\tau_1))/\dot{t}(\tau_1)^2$ as long as $\dot{t}(\tau) > 0$ for $\tau > \tau_1$. This strict concavity implies we must have $\dot{t}(\tau) = 0$ eventually.

4. An equation for timelike geodesics. Combining the integrability in (2) with the symmetry in (3), we arrive at the following description. Follow the arc (parametrized by proper time) considered in (2) (which starts at $(\bar{x}, \bar{t})$, where $\bar{x} < 0$ and $\dot{x}(\bar{\tau}) = 0$) until the next time $\dot{x}$ vanishes: $\dot{x}(\tau_0)$=0, where necessarily $t(\tau_0) = \bar{t} + \pi$ and $x(\tau_0) = -\bar{x} > 0$ (this can be seen by differentiating in τ the equation given next). This arc of γ from $\bar{\tau}$ to τ_0 has equation:

$$\cos(t-\bar{t}) = -\frac{C}{\sqrt{C^2-1}}\sin x, \quad C = \dot{t}(\bar{\tau})/\cos^2(\bar{x}).$$

The next arc of γ (from $(-\bar{x}, \bar{t}+\pi)$ to $(\bar{x}, \bar{t}+2\pi)$) is obtained by reflecting on the line $\{t = t(\tau_0) = \bar{t}+\pi\}$, and has equation:

$$\cos(t-\bar{t}-\pi) = \frac{C}{\sqrt{C^2-1}}\sin x,$$

the same equation as before. We conclude the image of the complete geodesic γ is given by the equation:

$$x = -\arcsin[\frac{\sqrt{C^2-1}}{C}\cos(t-\bar{t})], \quad t \in \mathbb{R}, \quad C = \dot{t}(\tau_0)/\cos^2(x(\tau_0))$$

(where $\tau_0 \in \mathbb{R}$ is arbitrary and $\bar{t} = t(\bar{\tau})$ for some $\bar{\tau}$ satisfying $\dot{x}(\bar{\tau}) = 0$ and $x(\bar{\tau}) < 0$). A consequence of this is the following conclusion:

Exercise 6: If γ is an arbitrary timelike geodesic, we have:

$$(\bar{x}, \bar{t}) \in \gamma(\mathbb{R}) \Rightarrow ((-1)^n\bar{x}, \bar{t}+n\pi) \in \gamma(\mathbb{R}),$$

for all $n \in \mathbb{Z}$.

The failure of 'timelike geodesic connectivity' for chronologically related points follows immediately from this. For example, all the points on the line $\{(\bar{x}, t); t \in \mathbb{R}\}$ are chronologically related to each other, but $(\bar{x}, t)$ is connected by geodesics only to the points $(\bar{x}, t+2n\pi)$, $n \in \mathbb{Z}$.

Exercise 7: Sketch the image of the exponential map from an arbitrary point $(\bar{x}, \bar{t})$ in this spacetime (or convince yourself that the picture on p.7 of Penrose's book is correct.)

5. Problem/speculation. How general is this behavior for timelike geodesics? Although it is somewhat 'pathological', the arguments seem fairly general, and should give a larger family of examples. (Say, for 'product' Lorentzian manifolds with rotationally symmetric spacelike slices?) The example has the following property:

Definition: A timelike-complete Lorentzian manifold (M, g) is 'totally conjugate' if for each $z_0 \in M$ and complete timelike geodesic γ through z_0, we may find points $z_1, z_{-1} \in \gamma(\mathbb{R})$ (with $z_{-1} << z_0 << z_1$) so that *any* timelike geodesic through z_0 also goes through z_1 and z_{-1}.

Problem: Find more examples of 'totally conjugate' Lorentzian manifolds. Particularly interesting would be examples where this property persists under small perturbations of the metric.

Remark. The example described here is 'causally simple'. On the other hand, any 'globally hyperbolic' spacetime is timelike geodesically connected. ('Globally hyperbolic' means 'strongly causal' and $J^+(p) \cap J^-(q)$ compact whenever $p << q$. This is stronger than 'causally simple', and indeed is a very strong global condition- for instance, it implies M is topologically a product $S \times \mathbb{R}$, where S is a spacelike hypersurface.)

References:

1. R. Penrose, *Techniques of differential topology in Relativity*, SIAM 1972 (p.7)

2. J.K.Beem, P.Ehrlich, K.L. Easley, *Global Lorentzian Geometry*, 2nd. ed., Marcel Dekker 1996 (cover and section 6.1).