WHITNEY'S IMMERSION / EMBEDDING THEOREMS [1]

Preliminaries 1: Linear Algebra. Denote by $M(m \times n; k)$ the space of $m \times n$ matrices with real coefficients of rank k. *Claim.* This is a smooth surface in $M_{m\times n}$, of dimension $k(m+n-k)$, and therefore codimension $(m-k)(n-k)$.

Write $X \in M_{m\times n}$ in block form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \in M_{k\times k}.$$

Let $W = \{X \in M_{m\times n}; \det A \neq 0\}$. We have:

$$U := W \cap M(m \times n; k) = \{X \in W; D = CA^{-1}B\},$$

since the rank of $X \in M_{m\times n}$ is the same as the rank of:

$$\begin{bmatrix} I_k & 0 \\ -CA^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix},$$

and hence equals k iff $D - CA^{-1}B$=0. Parametrize U by $\phi : U_0 \to U$,

$$U_0 = \{(A, B, C) \in M_{k\times k} \times M_{k\times(n-k)} \times M_{(m-k)\times k}\},$$

$$\phi(A, B, C) = \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}.$$

If $X \in M(m \times n; k)$ is arbitrary, we may find a C^∞ diffeomorphism h of $M_{m\times n}$ (defined by row/column exchanges) such that $h(X) \in U$.

Preliminaries 2: sets of measure zero in a manifold.

$X \subset \mathbb{R}^n$ has *measure zero* if $\forall \epsilon > 0$ one may find a countable covering $X \subset \bigcup_{i\geq 1} C_i$ by cubes s.t. $\sum_i vol(C_i) < \epsilon$.

A countable union of sets of measure zero has measure zero.

$meas(X) = 0$ iff for any $p \in X$ we have a neighborhood V of p in $\mathbb{R}^n$ s.t. $meas(V \cap X) = 0$;

If $meas(X) = 0$ and $f \in Lip_{loc}(X; \mathbb{R}^m)$, we have $meas(f(X)) = 0$. (*Exercise 1.*)

[1] Adapted from 'Variedades Diferenciáveis', by Elon L. Lima, IMPA monograph 1973 (unfortunately untranslated)

Definition: $X \subset M$ has measure zero in M if for any $p \in M$ there exists a chart (U, φ) at p s.t. $meas(\varphi(U \cap X)) = 0$ in $\mathbb{R}^n$.

Proposition 1: If $f : M^m \to \mathbb{R}^s$ is C^1 and $m < s$, $meas(f(M)) = 0$ in $\mathbb{R}^s$.

Proof. Given $p \in M$, by considering a chart (U, φ) at p we may assume there exist cubes $C = C' \times C'' \subset \mathbb{R}^s = \mathbb{R}^m \times \mathbb{R}^{m-s}$ and $\tilde{f} : C \to \mathbb{R}^s$ (Lipschitz) so that $f(U) = \tilde{f}(C' \times \{0\})$.

Proposition 2. Consider the following situation: $f \in C^1(M^m; \mathbb{R}^s)$, $N \subset \mathbb{R}^s$ a (smooth) surface of codimension greater than m. *Claim:* For every $v \in \mathbb{R}^s$ except for a set of measure zero, the translate of $f(M)$ by v misses N:

$$[f(M) + v] \cap N = \emptyset.$$

Indeed, define $\phi : M \times N \to \mathbb{R}^s$ by $\phi(p, q) = f(p) - q$. If the intersection is non-empty, $-v$ is in the image $\phi(M \times N)$. But since $dim(M) + dim(N) < s$, this image has measure zero in $\mathbb{R}^s$ (by prop. 1)

The same conclusion holds if instead of a single N we have $X = \bigcup_{i \geq 1} N_i$, where each N_i has codimension greater than m in $\mathbb{R}^s$.

3. Local versions of the immersion theorem in $\mathbb{R}^m$.

Proposition 3. Let $f : B_3^m \to \mathbb{R}^s$ be a C^r map, $r \geq 2$. Assume $s \geq 2m$. Then $\forall \epsilon > 0$ we may find an immersion $g : B_3^m \to \mathbb{R}^s$ with $|f - g|_{C^1(B_3)} < \epsilon$.

Proof. The set of $s \times m$ matrices of rank $i < m$ is a surface $N_i \subset M_{s \times m}$ of codimension $(m - i)(s - i) \geq 1.(2m - (m - 1)) = m + 1$.

Remark: Thus proposition 2 implies the density result for linear transformations: if $T \in M_{s \times m}$ with $s \geq 2m$, $T + A$ has rank m (that is, is a linear immersion $\mathbb{R}^m \to \mathbb{R}^s$) for all $A \in M_{s \times n}$ except for a set of measure zero in $M_{s \times n}$, in particular for A's of arbitrarily small norm.

Let $g(x) = f(x) + Ax$, for $A \in M_{s \times m}$. Then $dg(x) = df(x) + A$, and since $df \in C^1(B_3; M_{s \times m})$, proposition 2 implies $df + A$ has rank m for all A's except for a set of measure zero in $M_{s \times m}$, in particular for A's of arbitrarily small norm.

The perturbation can be localized: there exists $h \in C^r(B_3; \mathbb{R}^s)$ so that $|h - f|_{C^1(B_3)} < \epsilon$ and $h = f$ in $(B_2)^c$. Just let $h(x) = f(x) + \phi(x)(g(x) - f(x))$, with g as in the last paragraph and $\phi \equiv 1$ in B_1, $\phi \equiv 0$ off B_2.

Where f is already an immersion, it is possible to leave it unchanged (this will be used to globalize the result):

Proposition 4. In the setting of prop.3, assume $F \subset B_3$ is closed and $f_{|F}$ an immersion. Then we may find $h \in C^r(B_3; \mathbb{R}^s)$ immersion in $\bar{B}_1 \cup F$, with $|h - f|_{C^1(B_3)} < \epsilon$, satisfying $h = f$ in $F \cup (B_2)^c$.

Proof. Let $K = F \cap \bar{B}_2$, which is compact in B_3. It is clearly enough to prove the statement replacing F by K. Let $V \subset B_3$ be a relatively compact, open neighborhood of K, with the property that $f_{|\bar{V}}$ is still an immersion. Then:

$$\bar{B}_1 \cup K \subset (\bar{B}_1 - V) \cup \bar{V}.$$

Find ξ smooth, supported in B_3, so that $\xi \equiv 1$ in $\bar{B}_1 - V$ and $\xi \equiv 0$ in $K \cup (\bar{B}_3 - B_2)$, then set:

$$h = f + \xi(g - f),$$

where g is an immersion close to f in B_3 (as in prop. 3), indeed sufficiently close that $|h - f|_{C^1} < \epsilon$ in B_3, so that in particular h is still an immersion in $\bar{V}$ (by the 'simple fact' below in Exercise 2). Since, in addition, $h = g$ in $\bar{B}_1 - V$, we conclude h is an immersion in $\bar{B}_1 \cup K$. Finally, $h = f$ in $K \cup (B_2)^c$ since $\xi \equiv 0$ on this set.

4. The Whitney and compact-open topologies.

*Definitions.*We consider two topologies in spaces of C^r $(r \geq 1)$ maps from a manifold M^m to $\mathbb{R}^s$.

1) *Whitney C^1 topology.* Basic neighborhoods of $f \in C^r(M; \mathbb{R}^s)$ are defined by a choice of positive continuous function $\epsilon \in C^0(M; R_+)$:

$$W^1(f; \epsilon) := \{g \in C^r(M; \mathbb{R}^s); |f(p) - g(p)| < \epsilon(p), |df(p) - dg(p)| < \epsilon(p), \forall p \in M\}.$$

(Note: we assume M is endowed with a fixed Riemannian metric.) Although the resulting topology is not metrizable if M is non-compact, we abbreviate the two inequalities in the definition by the notation $|f - g|_{C^1} < \epsilon$ (here $|f - g|_{C^1}$ and ϵ are *functions* on M).

Remark: An equivalent topology is obtained by fixing a countable, locally finite covering $\mathcal{U} = \{U_i\}_{i \geq 1}$ by relatively compact open sets and considering sequences $a = (a_i)_{i \geq 1}$ of positive reals, taking as the associated basic neighborhood of f:

$$W^1(f; a) = \{g \in C^r(M; \mathbb{R}^s); |f - g|_{C^1(\bar{U}_i)} < a_i \forall i \geq 1\}.$$

2) *Compact-open topology.* This is the topology of uniform C^1 convergence on compact sets. A basic neighborhood of f is defined by choosing $K \subset M$ compact and $\delta > 0$:

$$V^1(f; K, \delta) = \{g \in C^r(M; \mathbb{R}^s); |f - g|_{C^1(K)} < \delta\}.$$

This topology has a countable basis and is metrizable. For compact manifolds both topologies coincide, but in the non-compact case the Whitney topology is finer: it has enough open sets that many natural classes of maps (immersions, submersions) are open.

Proposition 5. The class of C^r immersions $Imm^r(M; \mathbb{R}^s)$ is open in $C^r(M; \mathbb{R}^s)$, in the Whitney C^1 topology.

Proof. Note the following simple fact (*Exercise 2*): if $f \in C^1(U; \mathbb{R}^s)$ is an immersion (where $U \subset \mathbb{R}^m$) and $K \subset U$ is compact, there exists $\eta > 0$ s.t. if $g \in C^1(U; \mathbb{R}^s)$ and $|g - f|_{C^1(K)} < \eta$, then $g_{|K}$ is an immersion.

We use the definition of the Whitney topology via bases defined by sequences. Fix a countable, locally finite covering $\{U_i\}_{i \geq 1}$ by relatively compact domains of charts. Let $f \in C^r(M; \mathbb{R}^s)$ be an immersion. From the 'simple fact' in the last paragraph, we may find $a_i > 0$ so that if $|f - g|_{C^1(\bar{U}_i)} < a_i$, then $g_{|\bar{U}_i}$ is an immersion. Thus the Whitney C^1 basic neighborhood $W^1(f; a)$ with $a = (a_i)$ consists of immersions.

Remark. Openness of immersions is not true for the compact-open topology (as simple examples show.)

5. The Whitney immersion theorem.

Theorem. Assume $s \geq 2m$, $r \geq 2$ The class $Imm^r(M^m; \mathbb{R}^s)$ of immersions is dense in $C^r(M^m; \mathbb{R}^s)$, endowed with the Whitney C^1 topology.

This means: given $f \in C^r(M; \mathbb{R}^s)$ and $\epsilon \in C^0(M; \mathbb{R}_+)$, there exists an immersion $g \in C^r(M; \mathbb{R}^s)$ so that $|g - f|_{C^1} < \epsilon$ on M (as functions on M).

Introduce the following *global notation*: when we consider a locally finite open cover of M by precompact domains of open charts $(U_i, \phi_i)_{i \geq 1}$, we assume for each of these domains U the corresponding chart ϕ maps $U = U^{(3)}$ to $B_3 \subset \mathbb{R}^m$, and let $U^{(2)} = \phi^{-1}(B_2)$, $U^{(1)} = \phi^{-1}(B_1)$. We always assume the $U_i^{(1)}$ cover M.

Proof of theorem. Consider a cover (U_i) as just described. For $n \geq 1$, denote by $M_n \subset M$ the open, precompact submanifold of M consisting of the union of the first n $U_i^{(1)}$'s. Given f, assume f has been modified to a C^r map $f_n : M \to \mathbb{R}^s$, so that f_n is an immersion on $\bar{M}_n$ and $|f - f_i|_{C^1} < \frac{\epsilon}{2^i}$ as functions on $U_i^{(1)}$ for $i = 1, \cdots, n$. Let $U^{(3)}$ be the $(n+1)$th. domain; we'll modify f_n to f_{n+1}, with $f_{n+1} = f_n$ off $U^{(2)}$. By proposition 4, there exists $h \in C^r(\bar{U}^{(3)}; \mathbb{R}^s)$ with the properties: (i) h is an immersion in $\bar{U}^{(1)} \cup (\bar{M}_n \cap \bar{U}^{(3)})$ (since $h = f_n$ in $\bar{M}_n \cap \bar{U}^{(3)}$) ; (ii) $h = f_n$ in $\bar{U}^{(3)} - U^{(2)}$; (iii) $|h - f_n|_{C^1} < \frac{\epsilon}{2^{n+1}}$, as functions on $U^{(1)}$.

Thus if we define $f_{n+1} \in C^r(M; \mathbb{R}^s)$ by setting:

$$\begin{cases} f_{n+1} = h \text{ on } U^{(3)} \\ f_{n+1} = f_n \text{ on } M - U^{(2)}, \end{cases}$$

we have that f_{n+1} is an immersion on $M_{n+1} = M_n \cup U^{(1)}$, satisfying $|f_{n+1} - f_n|_{C^1} < \frac{\epsilon}{2^{n+1}}$, as functions on $U^{(1)}$- so we may continue to the next $U^{(3)}$ on the list.

To finish the proof, we set $g(p) = \lim f_n(p)$. This is a pointwise limit, but note that at each $p \in M$ this is a finite sequence (since each $p \in M$ is in only finitely many $U^{(1)}$'s). So in fact the limit is C^r-uniform on compact sets, and g is C^r, as well as an immersion. It is clear from the construction that $|f - g|_{C^1} < \sum_{n=1}^{\infty} |f_{n+1} - f_n|_{C^1} < \epsilon$ on M. QED

6. Density of injective immersions. First we consider a local stability result in $\mathbb{R}^m$ (similar to the 'simple fact' used in the proof of proposition 5.)

Proposition 6. Let $f \in C^1(U; \mathbb{R}^s)$, where $U \subset \mathbb{R}^m$ is open; let $K \subset U$ be compact and *convex;* assume $f_{|K}$ is an embedding. Then there exists $\eta > 0$ s.t. if $g \in C^1(U; \mathbb{R}^s)$ and $|f - g|_{C^1(K)} < \eta$, then $g_{|K}$ is an embedding (equivalently, an injective immersion).

Proof. By prop.5, we have $\eta_1 > 0$ s.t. $|f - g|_{C^1(K)} < \eta_1 \Rightarrow g$ is an immersion on K. So we have to work on 'injective on K'.

First, since f is a C^1 immersion on the compact set K, we may find $\delta > 0$ and $c > 0$ so that $|f(x) - f(y)| \geq c|x - y|$ if $x, y \in K$ and $|x - y| < \delta$. (*Exercise 3.*) Second, we may always pick $d > 0$ so that $|f(x) - f(y)| \geq d$ if $x, y \in K$ with $|x - y| \geq \delta$.

Let $h = f - g$; note we are assuming $|h|_{C^1(K)} < \eta$ in K, and since K is convex the mean-value inequality implies $|h(x) - h(y)| \leq \eta|x - y|$ for $x, y \in K$.

Thus if $x, y \in K$ with $|x - y| < \delta$ we have:

$$|g(x) - g(y)| \geq |f(x) - f(y)| - |h(x) - h(y)| \geq c|x - y| - \eta|x - y|,$$

while if $x, y \in K$ with $|x - y| \geq \delta$:

$$|g(x) - g(y)| \geq |f(x) - f(y)| - |h(x)| - |h(y)| \geq d - 2\eta.$$

Thus if we pick $\eta < \min\{\eta_1, c, \frac{d}{2}\}$ we guarantee g is an injective immersion on K.

Now for the global version.

Theorem. Assume $s \geq 2m + 1$. Then the class of C^r injective immersions is dense in the space of C^r maps from M^m to $\mathbb{R}^s$, with respect to the Whitney C^1 topology.

This means: given $f \in C^r(M; \mathbb{R}^s)$ and $\epsilon \in C^0(M; \mathbb{R}_+)$ we may find an injective immersion $g \in C^r(M; \mathbb{R}^s)$ so that $|g - f|_{C^1} < \epsilon$ (as functions on M).

Proof. We may assume f is an immersion, and that $|g - f|_{C^1} < \epsilon$ implies g is an immersion (and hence, locally an embedding). The issue is injectivity.

The proof is organized as that of the immersion theorem, so refer to *global notation* in that proof. We pick the $U_i = U_i^{(3)}$ as before, but making sure that if two of them intersect f is injective in their union, also assumed to be the domain of a chart (this is possible since f is locally an embedding). Now let M_n be the union of the first n $U_i^{(1)}$'s, and assume we have changed f to an immersion $f_n \in C^r(M; \mathbb{R}^s)$ which is injective in M_n. The goal is to change f_n by a small amount in the $(n+1)$th. $U^{(3)}$ so that the resulting map f_{n+1} coincides with f_n off $U^{(2)}$, and in addition satisfies (with $M_{n+1} := M_n \cup U^{(1)}$):

(i) f_{n+1} is injective in each $U_i^{(3)}$, and if two of the $U_i^{(3)}$'s intersect, f_{n+1} is injective in their union;

(ii) f_{n+1} is injective in M_{n+1}.

(iii) $|f_{n+1} - f_n|_{C^1} < \frac{\epsilon}{2^{n+1}}$, as functions on M.

The idea is to consider perturbations of f of the following form. Pick $\lambda \in C^k(M; [0, 1])$ (say M is $C^k, k \geq r$) so that $\lambda \equiv 1$ in $\bar{U}^{(1)}$, $\lambda \equiv 0$ off $U^{(2)}$ and also in $\bar{M}_n - U^{(1)}$. Then choose $0 \neq v \in \mathbb{R}^s$ and define $f_{n+1} \in C^r(M; \mathbb{R}^s)$ by:

$$f_{n+1}(p) = f_n(p) + \lambda(p)v.$$

Recall that in proposition 6 we saw that 'injectivity of immersions is stable on compact subsets of $\mathbb{R}^m$' [2], which certainly transfers to chart do-

[2]We should be careful about what informal statements like this one are really saying- consider the 'figure 6 example'

mains on M. (i) holds for f, and inductively for f_n, so (i) (and clearly also (iii)) will also for f_{n+1}, provided v is picked with sufficiently small norm. As for (ii), we do know that f_{n+1} is injective in $M_n - \bar{U}^{(3)}$ (since it equals f_n on this set, and f_n is injective on M_n). So the issue is ensuring that $f_{n+1}(p) \neq f_{n+1}(q)$ if $p \in M_n - \bar{U}^{(3)}$ and $q \in U^{(1)}$.

So we pick $v \in \mathbb{R}^s$ with small norm, and so that:

$$[f_n(U^{(3)}) + v] \cap f_n(M_n - \bar{U}^{(3)}) = \emptyset.$$

(Why can we do this? Refer to proposition 2, applied to the m-dimensional manifold $U^{(3)}$ and the m-dimensional surface $f_n(M_n - \bar{U}^{(3)})$ in $\mathbb{R}^s$, which has codimension $s - m$, strictly greater than m since $s \geq 2m + 1$.)

Then with p and q as above: $f_{n+1}(p) = f_n(p)$ and $f_{n+1}(q) = f_n(q) + v$, so they can't coincide. We conclude (iii) also holds, and we may proceed.

To finish the proof, let $g(p) = \lim f_n(p)$. This is a pointwise limit, but as noted earlier the sequence is finite at each $p \in M$, so in fact the convergence is C^r-uniform on compact sets and the limit g is a C^r immersion, and satisfies:

$$|g - f|_{C^1} \leq \sum_{n=1}^{\infty} |f_{n+1} - f_n| < \epsilon$$

(as functions). The limit is also injective, since given any two points $p \neq q$ in M we may find $n \geq 1$ so that $g(p) = f_n(p)$, $g(q) = f_n(q)$ and $p, q \in M_n$.QED

7. Whitney's embedding theorem.

Theorem. If $m \geq 2s+1$ and M^m is a C^k manifold, there exists a proper embedding of class C^k from M to $\mathbb{R}^s$.

Proof. First we find a proper map $f \in C^k(M; \mathbb{R}^s)$. Let $\sum_i \chi_i \equiv 1$ be a C^k partition of unity on M. Then $\lambda(p) := \sum_{i \geq 1} i\chi_i(p)$ defines a proper C^k function on M, and if we pick $0 \neq v \in \mathbb{R}^s$ arbitrarily and set $f(p) := \lambda(p)v$, f is a C^k proper map. *(Verify this as Exercise 4.)*

By the previous theorem, we may find an injective immersion $g \in C^k(M; \mathbb{R}^s)$ so that $|f - g|_{C^1} < 1$ on M. Hence their limit sets coincide, and are both empty: $L(f) = L(g) = \emptyset$; so g is also proper (and therefore is an embedding.) QED

Remark 1. In fact we have the following statement: if $Prop^r(M; \mathbb{R}^s)$ denotes the space of proper C^r maps, then $Emb^r(M; \mathbb{R}^s)$ (the space of proper

C^r embeddings) is dense in $Prop^r(M; \mathbb{R}^s)$, with respect to the Whitney C^1 topology. Right?

Remark 2. It is *not* true that $Emb^r(M; \mathbb{R}^s)$ is dense in $C^r(M; \mathbb{R}^s)$ (with the Whitney C^1 topology), and for a very interesting reason: injective immersions $\mathbb{R} \to \mathbb{R}^3$ in $\mathbb{R}^3$ may exhibit non-trivial recurrence (that is, $f(x) \in L(f)$ for some $x \in \mathbb{R}$), C^1-stably.

In preparation for this example, consider the following exercise:

Exercise: (i) Let $f : M \to \mathbb{R}^s$ be a C^1 injective immersion. Recall the limit set of f is defined (for M non-compact) as:

$$L(f) = \{X \in \mathbb{R}^s; X = \lim x_k \text{ for some sequence } x_k \to \infty\}.$$

Here $x_k \to \infty$ for a sequence $(x_k) \subset M$ means x_k has no accumulation point in M, equivalently: given $K \subset M$ compact, we can always find $k_0 \geq 1$ so that $x_k \in M - K$ for $k \geq k_0$.

Claim. If $f(M) \cap L(f) \neq \emptyset$, f is not an embedding into $\mathbb{R}^s$.

(ii) Give an example to show that two injective immersions $f, g : \mathbb{R} \to \mathbb{R}^2$ may satisfy $|f(x) - g(y)| < C$ for all t (and some fixed $C > 0$), but have disjoint limit sets (both non-empty). Is this possible if, say, $L(f) = \emptyset$?

Idea. Consider a curve spiraling onto the unit circle and contained in a vertical strip of width 3, and translate the picture by 3 units. For the second part, find a productive statement equivalent to $L(f) = \emptyset$.

(iii) Say that a continuous positive function $\epsilon \in C^0(M; \mathbb{R}_+)$ satisfies $\epsilon \to 0$ at infinity if $(\forall \delta > 0)(\exists K \subset M \text{ compact})$ $(x \in M - K \Rightarrow 0 < \epsilon(x) < \delta)$.

Claim. If $|f - g| < \epsilon$ (as functions on M) for a function $\epsilon \in C^0(M, \mathbb{R}_+)$ tending to 0 at infinity, then $L(f) = L(g)$. (Here $f, g \in C^1(M; \mathbb{R}^s)$ are C^1 injective immersions.)