

PARTITIONS OF UNITY and RIEMANNIAN METRICS

1. General Topology. Recall the hierarchy of *separation properties* for a general topological space X :

- (a) X is *Hausdorff* if open sets separate points;
- (b) A Hausdorff space X is *regular* if open sets separate points and closed sets; equivalently, if given $x \in X$ and U open neighborhood of x , we may find V open so that $x \in v \subset \bar{V} \subset U$.
- (b1) A Hausdorff space X is *completely regular* if given any $x \in X$, U open neighborhood of X , we may find $f : X \rightarrow [0, 1]$ continuous so that $f(x) = 1$, $\text{supprt}(f) \subset U$.
- (c) A Hausdorff space X is *normal* if open sets separate closed sets; equivalently, if given $C \subset X$ closed and an open neighborhood U of C , we may find $V \subset U$ open so that $C \subset V \subset \bar{V} \subset U$.

Theorem (Urysohn's lemma), [D, p.146]. A Hausdorff space X is normal iff given $C \subset X$ closed, $U \subset X$ open neighborhood of A , we may find a continuous function $f : X \rightarrow [0, 1]$ so that $f \equiv 1$ on C , $f \equiv 0$ on $X \setminus U$.

Every subspace of a normal space is completely regular (but not necessarily normal); every completely regular space is regular. A completely regular space need not be normal!

Locally compact spaces. A Hausdorff space X is *locally compact* if any point has an open neighborhood with compact closure; equivalently ([D, p.238]) (i) for each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \bar{V} \subset U$; or (ii) X has a basis consisting of relatively compact open sets.

Every locally compact Hausdorff space is completely regular. ([D, p.238]). If X is second countable and locally compact, it has a *countable* basis consisting of relatively compact open sets. ([D, p.238]).

Covering properties. A Hausdorff space X is *paracompact* if every open covering admits a locally finite refinement.

Every paracompact space is normal ([D, p.163]).

Lindelöf's theorem Any second countable space is *Lindelöf*: every open covering has a countable subcovering. [D, p.174]

Morita's theorem A regular Lindelöf space is paracompact ([D, p.174])

Thus, a second countable Hausdorff, locally compact space is paracompact.

Let X be paracompact. Then for each covering $\{U_\alpha\}_{\alpha \in A}$ of X , there

exists a subordinate partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ ($\varphi_\alpha : X \rightarrow [0, 1]$ continuous, $\text{support}(\varphi_\alpha) \subset U_\alpha$, $\sum_\alpha \varphi_\alpha \equiv 1$.) ([D, p 170].

Def. A locally compact space X is σ -compact if it can be expressed as the union of countable many compact spaces. Equivalently: X is a locally compact Lindelöf space ([D. p. 241]).

Urysohn's metrization theorem. If X is a normal Hausdorff space with countable basis, then X is metrizable. ([D, p.195])

In particular, if X is Hausdorff, locally compact, and second countable, then X is metrizable.

Any metric space is paracompact (Stone's theorem) ([K, p.160]).

If X is a connected, locally compact metric space, then X is σ -compact ([S ,p.4]).