SOME MULTILINEAR ALGEBRA

1. Tensor products. Let V, W be vector spaces (over $\mathbb{R}$, say). Let $F(V, W)$ be the free vector space over $\mathbb{R}$: its elements are formal finite linear combinations of points (v, w) with coefficients in $\mathbb{R}$ (this means: functions $V \times W \to \mathbb{R}$, vanishing at all but finitely many ordered pairs). $F(V, W)$ has a natural vector space structure (over $\mathbb{R}$.) Ler $R(V, W)$ be the subspace spanned by all elements of the form:

$$\lambda(v, w) - (\lambda v, w) \quad \lambda(v, w) - (v, \lambda w)$$

$$(v, w_1 + w_2) - (v, w_1) - (v, w_2) \quad (v_1 + v_2, w) - (v_1, w) - (v_2, w),$$

for all $\lambda \in \mathbb{R}, v, v_1, v_2 \in V, w, w_1, w_2 \in W$.

The quotient space $F(V, W)/R(V, W)$ is called the *tensor product* of V and W (denoted $V \otimes W$). The projection of (v, w) is denoted $v \otimes w$, and in the quotient vector space structure we have:

$$(\lambda v_1 + v_2) \otimes w = \lambda v_1 \otimes w + (v_2 \otimes w), \quad v \otimes (\lambda w_1 + w_2) = \lambda v \otimes w_1 + v \otimes w_2.$$

Tensor products have the following useful property: if $T : V \times W \to E$ is a bilinear map to a vector space E (that is, linear in each argument with the other argument held fixed), there is a unique linear map $L : V \otimes V \to E$ so that $T(v, w) = L(v \otimes w)$. Conversely, given L, this formula defines a bilinear map T.

Assume V and W are finite-dimensional. Let V^* be the dual of V (the vector space of linear functionals $V \to \mathbb{R}$). There is a natural isomorphism $V^* \otimes W \to \mathcal{L}(V; W)$, taking each element of the form $\theta \otimes w$ to the linear map $T(v) = \theta(v)w$. This shows that $dim(V \otimes W) = dim(V)dim(W)$. Indeed if $(v_i)_i, (w_j)_j$ are bases for V, W (resp.), $(v_i \otimes w_j)_{i,j}$ is a basis for $V \otimes W$.

Analogously (or iterating the construction) we may define the r-fold tensor product $V \otimes \ldots \otimes V$ (r factors), corresponding to r-multilinear maps $V \times \ldots \times V \to W$. The space of tensors of type (r, s) on V is:

$$V^* \otimes \ldots \otimes V^* \otimes V \otimes \ldots \otimes V \quad (r \text{ factors } V^*, s \text{ factors } V),$$

wihch is isomorphic to $\mathcal{L}(V \otimes \ldots \otimes V; V \otimes \ldots V)$ (r factors on the left, s on the right).

2. Exterior algebras. We define the space $\Lambda_2(V)$ of *alternating 2-vectors* as the quotient space of $V \otimes V$ by the subspace spanned by $\{v \otimes v; v \in V\}$. Thus, in $\Lambda_2(V)$, the equivalence class $v \wedge w$ of $v \otimes w$ satisfies: $v \wedge w = -w \wedge v$. The space $\Lambda_r(V)$ of alternating r-vectors is defined analogously.

A 2-linear map $T : V \times V \to W$ is *alternating* if $T(v,w) = -T(w,v)$ (equivalently, if $T(v,v) = 0$.) An alternating 2-linear map corresponds naturally to a linear map $L : \Lambda_2(V) \to W$, via $L(v \wedge w) = T(v,w)$. Alternating p-linear maps are defined analogously, and correspond to linear maps $\Lambda_r(V) \to W$.

If $\{v_1, \ldots v_p\}$, $\{w_1, \ldots w_p\}$ are oriented bases of a p-dimensional subspace $E \subset V$, then:

$$w_i = \sum_j t_{ij} v_j, T = (t_{ij}) \text{ nonsingular } \Rightarrow w_1 \wedge \ldots \wedge w_p = (\det T) v_1 \wedge \ldots \wedge w_p.$$

This can be made more concrete if V has an inner product and $\{e_i\}$ is an orthonormal basis of V. The inner product on V induces one on $\Lambda_p(V)$ via:

$$\langle v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle).$$

Thus from $w_i = \sum_j \langle w_i, e_j \rangle e_j$ follows:

$$w_1 \wedge \ldots \wedge w_p = \langle w_1 \wedge \ldots \wedge w_p, e_1 \wedge \ldots \wedge e_p \rangle e_1 \wedge \ldots \wedge e_p.$$

By choosing an oriented basis, we may associate a simple element of $\Lambda_p(V)$ to each p-dimensional subspace of V; and conversely, equivalence classes of simple elements of $\Lambda_p(V)$ (modulo multiplication by positive scalars) define oriented p-dimensional subspaces of V (that is, define an element of the *oriented Grassmannian* $G_p(V)$).

3. Determinants. A linear map $T : V \to W$ induces in a natural way maps $T_r : \Lambda_r(V) \to \Lambda_r(W)$:

$$T_r(v_1 \wedge \ldots \wedge v_r) = (Tv_1) \wedge \ldots \wedge (Tv_r)$$

(extended linearly). In particular if $\dim(V) = \dim(W) = n$, $\Lambda_n(V)$ and $\Lambda_n(W)$ are one-dimensional, and T_n will be multiplication by a constant, the *determinant* of T.

If M and N are oriented n-manifold with volume n-forms Ω_M, Ω_N (resp.), the determinant of $Df : TM \to TN$ is defined via pullback:

$$f^* \Omega_N(p) = \det(Df_p) \Omega_M(p).$$

For instance, if $M^n \subset \mathbb{R}^{n+1}$ is a hypersurface oriented by the unit normal (Gauss map) $N : M \to S^n$, the induced volume form on M is given by:

$$\Omega_M(v_1, \dots, v_n)(p) = det_{n+1}[v_1|\dots|v_n|N], \quad v_i \in T_pM \subset \mathbb{R}^{n+1}.$$

(the determinant of a matrix given by its column vectors.) The volume form Ω_S on S^n is defined by exactly the same formula:

$$\Omega_S(w_1, \dots, w_n)(N) = det_{n+1}[w_1|\dots|w_n|N], \quad w_i \in T_NS^n \subset \mathbb{R}^{n+1}.$$

Thus the Jacobian of the Gauss map is given by:

$$\det(DN)(p) = det_{n+1}[DN(p)v_1|\dots|DN(p)v_n|N(p)].$$

In terms of the *shape operator* $S_p = -DN(p) : T_pM \to T_pM$ (note $T_pM = T_{N(p)}S^n$ as subspaces of $\mathbb{R}^{n+1}$):

$$\det(DN)(p) = (-1)^n det_{n+1}[S_pv_1|\dots|S_pv_n|N(p)].$$

Thus if $\{v_1, \dots, v_n\}$ is an orthonormal basis of T_pM diagonalizing S_p with eigenvalues $\lambda_1, \dots, \lambda_n$, we have for the Jacobian of the Gauss map:

$$\det(DN)(p) = (-1)^n \lambda_1 \dots \lambda_n = (-1)^n K_p,$$

where K_p is the *Gauss curvature* of M at p.