THE RIEMANN CURVATURE TENSOR, JACOBI FIELDS and GRAPHS

1. Geodesic deviation: the curvature tensor and Jacobi fields. Let (M, g) be a Riemannian manifold, $p \in M$. Suppose we want to measure the "instantaneous spreading rate" of geodesic rays issuing from p. The natural way to do this is to consider a geodesic variation:

$$f(t,s) = \exp_p(tv(s)), \quad v(s) \in T_pM, \quad v(0) = v, v'(0) = w.$$

Then the "spreading rate" is measured by the variation vector field $V(t)$:

$$V(t) = \frac{\partial f}{\partial s}_{|s=0} = d\exp_p(tv)[tw],$$

a vector field along the geodesic $\gamma(t) = \exp_p(tv)$. Let's try to find a differential equation satisfied by V. We have:

$$\frac{DV}{\partial t} = \frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}, \quad \frac{D^2V}{\partial t^2} = \frac{D}{\partial t}\frac{D}{\partial s}(\frac{\partial f}{\partial t}).$$

Setting $W = \partial_t f$ (a vector field along f), we see that $DW/\partial t \equiv 0$:

$$\frac{D}{\partial t}(\frac{\partial f}{\partial t}) = \frac{D}{\partial t}(d\exp_p(tv(s))[v(s)] = \frac{D}{\partial t}\dot{\gamma_s}(t) = 0,$$

since $\gamma_s(t) = \exp_p(tv(s))$ is a geodesic ($\gamma_s(0) = p, \dot{\gamma_s}(0) = v(s)$). Thus we need to compute the vector field $X(t)$ along f:

$$X(t) := \frac{D}{\partial t}\frac{DW}{\partial s} - \frac{D}{\partial s}\frac{DW}{\partial t},$$

for then, along $\gamma(t)$:

$$\frac{D^2V}{dt^2} = X(t).$$

We compute in a coordinate chart:

$$f(s,t) = (x^i(s,t)) \in \mathbb{R}^n, \quad W(s,t) = a^i(s,t)\partial_{x_i}.$$

Using the symmetry of the connection, we find:

$$X = a^i \partial_t x_k \partial_s x_j (\nabla_{\partial_{x_k}} \nabla_{\partial_{x_j}} \partial_{x_i} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} \partial_{x_i}).$$

We now consider the linearity over smooth functions of this commutator of covariant derivatives. We find:

$$\nabla_{\partial_{x_k}} \nabla_{\partial_{x_j}} (f\partial_{x_i}) - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} (f\partial_{x_i}) = f(\nabla_{\partial_{x^k}} \nabla_{\partial_{x_j}} \partial_{x_i} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} \partial_{x_i}),$$

and, assuming we have normal coordinates at p (so $\nabla_{\partial_{x_k}} \partial_{x_i} = 0$ at p):

$$\nabla_{f\partial_{x_k}} \nabla_{\partial_{x_j}} \partial_{x_i} - \nabla_{\partial_{x_j}} \nabla_{f\partial_{x_k}} \partial_{x_i} = f(\nabla_{\partial_{x_k}} \nabla_{\partial_{x_j}} \partial_{x_i} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} \partial_{x_i}).$$

Thus, by linearity, we have:

$$X(t) = \nabla_{\partial_t f} \nabla_{\partial_s f} W - \nabla_{\partial_t f} \nabla_{\partial_s f} W.$$

This suggests considering, given three vector fields X, Y, W, the vector field:

$$\nabla_X \nabla_Y W - \nabla_Y \nabla_X W.$$

A natural question is whether this is "tensorial" (linear over smooth functions) in each of X, Y, W. Starting with W, we find:

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)W + [X, Y]f.$$

This suggests subtracting the term $\nabla_{[X,Y]} W$. Computing again:

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})W.$$

This motivates the definition:

Definition. The $(3,1)$-*Riemann curvature tensor* R assigns to three vector fields (X, Y, Z) on M the vector field:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Exercise. This assignment is linear over smooth functions in each of X, Y and Z.

For a vector field $W(t, s)$ along an immersion $f(t, s)$, we get the *Ricci equation* :

$$\frac{D}{\partial t} \frac{D}{\partial s} W - \frac{D}{\partial s} \frac{D}{\partial t} W = R(\partial_t f, \partial_s f)W.$$

Returning to the vector field $W(t, s) = \partial_t f$ along the geodesic variation $f(t, s)$ (where the curves $t \mapsto f(t, s)$ are geodesics), we find:

$$\frac{D^2 V}{\partial t^2} = \frac{D}{\partial t} \frac{DW}{\partial s} = R(\partial_t f, \partial_s f)\partial_t f,$$

and at $s = 0$ (since $\partial_t f_{|s=0} = \dot{\gamma}$ and $\partial_s f_{|s=0} = V$):

$$\frac{D^2 V}{dt^2} + R(V, \dot{\gamma})\dot{\gamma} = 0.$$

This is the *Jacobi equation* for the "geodesic deviation" vector field $V(t)$; its solutions are *Jacobi fields* along $\gamma(t)$.

Remark. To find the first-order initial condition for $V(t)$, consider:

$$\frac{DV}{dt}_{|t=0} = \frac{D}{\partial t}\frac{\partial f}{\partial s}_{|t=0,s=0} = \frac{D}{\partial s}\frac{\partial f}{\partial t}_{|t=0,s=0} = \frac{d}{ds}\dot{\gamma}_s(0) = v'(0) = w.$$

We conclude:

$J(t) = d\exp_p(tv)[tw]$ is the Jacobi field along $\gamma(t) = \exp_p(tv)$ with IC $J(0) = 0, \dot{J}(0) = w$.

In particular: $d\exp_p(v)[w] = J(1)$. This expresses the differential of the exponential map in terms of the solution of a differential equation along $\gamma(t)$.

2. The case of graphs in euclidean space. Consider the surface $M \subset \mathbb{R}^{n+1}$:

$$M = graph(F) = \{(x, F(x)); x \in \mathbb{R}^n\}, \quad F : \mathbb{R}^n \to \mathbb{R}, \quad \nabla F(0) = 0.$$

The induced metric and inverse metric tensors are (check!):

$$g_{ij} = \delta_{ij} + F_i F_j, \quad g^{ij} = \delta^{ij} - \frac{F_i F_j}{1 + |\nabla F|^2}.$$

Given the assumptions made at $x = 0$, we find:

$$g_{ij|l}(0) = 0 \text{ and hence } \Gamma^k_{ij}(0) = 0.$$

For the first derivatives of the Christoffel symbols at $x = 0$:

$$\partial_{x_m}\Gamma^k_{ij}(0) = \frac{1}{2}(g_{ik|j|m} + g_{jk|i|m} - g_{ij|k|m}),$$

and for the curvature tensor[1]:

$$R(\partial_{x_j}, \partial_{x_k})\partial_{x_i} = \nabla_{x_j}\nabla_{x_k}\partial_{x_i} - \nabla_{x_k}\nabla_{x_j}\partial_{x_i}$$

$$= (\Gamma^l_{ik|j} - \Gamma^l_{ij|k} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{ij}\Gamma^l_{km})\partial_{x_l}$$

$$:= R^l_{jki}\partial_{x_l}.$$

[1]Note here we're using the notation R for what is in fact the pull-back $\varphi^* R$ of the (3,1) curvature tensor under the graph chart $\varphi(x) = (x, F(x))$

Thus at $x = 0$ we have:

$$R^l_{jki}(0) = \Gamma^l_{ik|j} - \Gamma^l_{ij|k}$$

Suppose we choose the axes so that the Hessian of F is diagonal at $x = 0$:

$$Hess(F)_{|0} = diag(\lambda_1, \ldots, \lambda_n), \quad F_{ij}(0) = \lambda_i \delta_{ij} \text{ (no sum)}.$$

Then, since $g_{ij|k|l}(0) = F_{ik}F_{jl}(0) + F_{il}F_{jk}(0) = \lambda_i\lambda_j(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$, the only non-zero second derivatives of the metric at $x = 0$ are:

$$g_{ii|i|i} = 2\lambda_i^2, \quad g_{ij|i|j} = g_{ij|j|i} = \lambda_i\lambda_j \quad (i \neq j).$$

(in particular, $g_{ii|j|j} = 0$ for $i \neq j$.) Thus the only potentially non-vanishing first-order derivatives of Christoffel symbols (at $x = 0$) have either all four indices equal, or two pairs of equal indices:

$$\Gamma^i_{ii|i} = \lambda_i^2,$$

$$\Gamma^i_{ij|j} = \frac{1}{2}(g_{ii|j|j} + g_{ij|i|j} - g_{ij|i|j}) = 0 \quad (i \neq j),$$

$$\Gamma^j_{ii|j} = \frac{1}{2}(g_{ij|i|j} + g_{ij|i|j} - g_{ii|j|j}) = \lambda_i\lambda_j, \quad (i \neq j).$$

We conclude:

$$\Gamma^l_{ik|j} = \lambda_i\lambda_j\delta_{ik}\delta_{lj},$$

and for the components of the curvature tensor at $x = 0$:

$$R^l_{jki} = \Gamma^l_{ik|j} - \Gamma^l_{ij|k} = \lambda_j\lambda_k(\delta_{ik}\delta_{lj} - \delta_{ij}\delta_{lk}) \quad \text{(no sum)}.$$

For the components of the (4,0)-curvature tensor:

$$R_{jkim} := \langle R(\partial_{x_j}, \partial_{x_k})\partial_{x_i}, \partial_{x_m}\rangle,$$

using the fact that $g_{ij}(0) = \delta_{ij}$ we find, at $x = 0$:

$$R_{jkim} = \lambda_j\lambda_k(\delta_{ik}\delta_{jm} - \delta_{ij}\delta_{km})$$

$$= D^2F(\partial_{x_i}, \partial_{x_k})D^2F(\partial_{x_j}, \partial_{x_m}) - D^2F(\partial_{x_i}, \partial_{x_j})D^2F(\partial_{x_k}, \partial_{x_m}),$$

where D^2F is the Hessian quadratic form of F.

By linearity, we have for arbitrary vector fields X, Y, Z, W on $\mathbb{R}^n$, and at $x = 0$[2]:

$$\langle(\varphi^* R)(X, Y)Z, W\rangle_g = D^2F(Z, Y)D^2F(X, W) - D^2F(Z, X)D^2F(Y, W).$$

Since both sides of this equation are "tensorial" (4-linear over functions), it in fact holds everywhere,and expresses the (4,0) curvature tensor in terms of the Hessian of F.

Kulkarni-Nomizu product. Given two quadratic forms $Q, \bar{Q}$ (i.e., symmetric bilinear forms) in a vector space E, their Kulkarni-Nomizu product is the 4-linear form on E:

$$(Q \odot \bar{Q})(x, y, z, w) := \frac{1}{2}[Q(x, z)\bar{Q}(y, w) - Q(y, z)\bar{Q}(x, w) + \bar{Q}(x, z)Q(y, w) - \bar{Q}(y, z)Q(x, w)].$$

Exercise. (i) $Q \odot \bar{Q}$ has the same algebraic symmetries as the (4,0)-Riemann curvature tensor, except for the first Bianchi identity: it is skew-symmetric in (x, y), skew-symmetric in (z, w) and *symmetric* under swapping the ordered pairs (x, y) and (z, w). Thus $Q \odot \bar{Q}$ is a quadratic form in the space of alternating 2-vectors $\Lambda_2(E)$.

(ii) If $Q = \bar{Q}$, the 4-linear form $\omega = Q \odot Q$ also satisfies the algebraic Bianchi identity:

$$\omega(x, y, z, w) + \omega(y, x, z, w) + \omega(z, y, x, w) = 0.$$

In terms of the K-N product, the (4,0)-Riemann curvature tensor *Riem* of a graph has the expression:

$$Riem = -(D^2F \circ D\pi) \odot (D^2F \circ D\pi).$$

where $D\pi(p) : T_pM \to \mathbb{R}^n$ is the inverse of the differential graph chart $D\varphi$, and

$$(D^2F \circ D\pi)(X, Y) := D^2F(D\pi[X], D\pi[Y]), \quad X, Y \in T_pM$$

Recall that if E has an inner product, there is an associated inner product in $\Lambda_2(E)$ uniquely determined by:

$$\langle x \wedge y, z \wedge w\rangle = \langle x, z\rangle\langle y, w\rangle - \langle y, z\rangle\langle x, w\rangle$$

[2]Here we revert to more precise notation: $\varphi^* R$ is a (3,1) tensor on $\mathbb{R}^n$, the pullback of the curvature tensor under the graph chart φ

(which itself has the structure of a K-N product!) Note, in particular:

$$|x \wedge y|^2 = |x|^2|y|^2 - \langle x, y\rangle^2.$$

Thus, given a quadratic form on $\Lambda_2(E)$ we have an associated symmetric linear operator from $\Lambda_2(E)$ to $\Lambda_2(E)$. In the case of the (4,0)-Riemann curvature tensor *Riem*, the associated symmetric linear operator (at each $p \in M$:

$$\mathcal{R}_p : \Lambda_2(T_pM) \to \Lambda_2(T_pM)$$

is known as the *curvature operator*, and has found important applications in recent years.

The two-dimensional case. If $n = 2$, the only non-zero components of *Riem* (at $p \in M$) have the form $\langle R(X,Y)X,Y\rangle$, with X, Y linearly independent (in T_pM). If M is the graph of F:

$$\langle R(X,Y)X,Y\rangle = -(D^2F \odot D^2F)(D\pi X, D\pi Y, D\pi X, D\pi Y).$$

If the $\{e_1, e_2\}$ is an orthonormal basis of $(\mathbb{R}^n, g)$ diagonalizing D^2F with eigenvalues λ_1, λ_2, we find:

$$(D^2F \odot D^2F)(e_1, e_2, e_1, e_2) = D^2F(e_1, e_1)D^2F(e_2, e_2) - (D^2F(e_1, e_2))^2 = \lambda_1\lambda_2.$$

This motivates the definition, for general dimensions n (changing the order of the second pair X, Y to get rid of the sign):

Definition. Let $v, w \in T_pM$ be linearly independent. The *sectional curvature* of M along the 2-dimensional subspace $E \subset T_pM$ spanned by v and w is the real number $\sigma_E(p)$ defined by:

$$\langle R(X,Y)Y,X\rangle(p) = \sigma_E(p)|v \wedge w|^2_{g_p},$$

where X, Y are vector fields on M with $X(p) = v, Y(p) = w$ and $|v \wedge w|^2_{g_p} = |v|^2_{g_p}|w|^2_{g_p} - \langle v, w\rangle^2_{g_p}$.

Exercise. $\sigma_E(p)$ depends only on the two-dimensional subspace E, not on the choice of basis.

Thus, in the two-dimensional case (for the graph of a function F), the sectional curvature σ is the product of the eigenvalues of the Hessian D^2F ($\sigma = \lambda_1\lambda_2$), and explicitly determines the (4,0)-Riemann curvature tensor, via:

$$\langle R(X,Y)Z,W\rangle = -\sigma(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle) = -\sigma\langle X \wedge Y, Z \wedge W\rangle.$$

It also follows that:

$$\sigma > 0 \Leftrightarrow \lambda_1, \lambda_2 \text{ have the same sign.}$$

3. Hypersurfaces in euclidean space. Let $M^n \subset \mathbb{R}^{n+1}$ be a submanifold. Given $p \in M$, we define a local graph chart at p, $\varphi : U \to M$, $U \subset \mathbb{R}^n$ open, via:

$$\varphi(x) = (x, F(x)) \in U \times \mathbb{R}, \quad F : U \to \mathbb{R} \text{ smooth }, (0, F(0)) = p, \nabla F(0) = 0.$$

The "upward" unit normal in a neighborhood of p isgiven in this chart by the map:

$$N : U \to \mathbb{R}^{n+1}, \quad N(x) = \frac{(-\nabla F(x), 1)}{\sqrt{1 + |\nabla F(x)|^2}}.$$

Thus if $\hat{N} : M \to S^n$ denotes the *Gauss map* of M, we have:

$$\hat{N}(\varphi(x)) = N(x), \quad x \in U.$$

Vector fields X, Y in U correspond via φ to tangent vector fields $\bar{X}, \bar{Y} \in \chi_M$:

$$\bar{X} = D\varphi[X] = (X, \nabla F \cdot X), \quad \bar{Y} = D\varphi[Y] = (Y, \nabla F \cdot Y).$$

By direct computation at $x = 0$, we find:

$$\langle DN(0)[X], \bar{Y} \rangle = -D^2 F(0)(X, Y).$$

And the chain rule gives: $D\hat{N}(p)[\bar{X}] = D\hat{N}(p) D\varphi(0)[X] = DN(0)[X]$, so we find for the differential of the Gauss map:

$$\langle D\hat{N}(p)[\bar{X}], \bar{Y} \rangle = -D^2 F(0)(X, Y) = -D^2 F(0)(D\pi(p)[\bar{X}], D\pi(p)[\bar{Y}]),$$

where $D\pi(p) : T_p M \to \mathbb{R}^n, D\pi(p)[\bar{X}] = X$ if $\bar{X} = (X, \nabla F \cdot X)$.

Note that since the last equality is "tensorial" (bilinear in $\bar{X}, \bar{Y}$ over smooth functions), it in fact holds at all points of M. In addition, it shows that the left-hand side is symmetric in $(\bar{X}, \bar{Y})$ (since the right-hand side is). This leads to the important definition of the *second fundamental form* (of M in $\mathbb{R}^{n+1}$), the quadratic form on TM given in tems of the Gauss map $\hat{N}$ by:

$$A(\bar{X}, \bar{Y}) := -\langle D\hat{N}[\bar{X}], \bar{Y} \rangle.$$

Remarks. (i) The "first fundamental form" is the induced metric. (ii) The purpose of the negative sign is to make the sectional curvature of the graph of a convex function positive (see below).

Above we established that, for a graph:

$$A(\bar{X}, \bar{Y}) = D^2F(D\pi[\bar{X}], D\pi[\bar{Y}]),$$

where $D\pi$ is the inverse differential of the graph chart, $D\varphi$. Thus we have, for the (4,0)-curvature tensor of a hypersurface in $\mathbb{R}^{n+1}$ the beautiful relation:

$$Riem = -A \odot A.$$

To make this more concrete, consider a two-dimensional subspace $E \subset T_pM$ which is *invariant* under the second fundamental form. This means $S_p(E) \subset E$, where $S_p : T_pM \to T_pM$ is the self-adjoint operator (with respect to the induced metric at $p \in M$) associated with A_p. (Note $S_p = -D\hat{N}(p)$). Let $\{e_1, e_2\}$ be an orthonormal basis of E diagonalizing the restriction $S_{p|E}$, with $S_p(e_i) = \lambda_i e_i$ for $i = 1, 2$. We have for the sectional curvature of E:

$$\sigma_E = \langle R_p(e_1, e_2)e_2, e_1\rangle = A_p \odot A_p(e_1, e_2, e_1, e_2)$$

$$= A_p(e_1, e_1)A_p(e_2, e_2) - (A_p(e_1, e_2))^2 = \lambda_1\lambda_2 = \det(S_{p|E}).$$

This is an important conclusion: if E is a two-dimensional subspace of T_pM invariant under the "shape operator" $S_p = -D\hat{N}(p)$ at a point $p \in M$, the sectional curvature at p along E (which depends only on the first fundamental form and its derivatives up to second order) equals the determinant of the restriction of the shape operator to E (which seems to depend on the second fundamental form, or on the unit normal and its first derivative).

In two dimensions, the "invariance" condition is unnecessary. The eigenvalues of S_p are the "principal curvatures" at $p \in M$, and their product is the *Gauss curvature* $K = \lambda_1\lambda_2$ at p. We conclude:

Gauss's Teorema Egregium: $\sigma = K$.

The fact that the sectional curvature equals the Gauss curvature is surprising since σ depends only on the induced metric (tangential information), while K seems to depend on the embedding of the surface in $\mathbb{R}^3$ (specifically, on how the unit normal "turns" near p).

Recall also that in two dimensions the Gauss curvature of a hypersurface equals the Jacobian of the Gauss map, so we have:

$$\sigma(p) = K(p) = \det D\hat{N}(p), \quad p \in M.$$

4. The differential Bianchi identity.

Theorem. The (3,1) curvature tensor satisfies:

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.$$

Proof. We compute in a geodesic frame at P, so that $\nabla_X Y(p) = 0$ for all vector fields X, Y. Then the left-hand side is, at p (using $[X,Y](p) = 0$):

$$\nabla_X([\nabla_Y, \nabla_Z]W) + \nabla_Y([\nabla_Z, \nabla_X]W) + \nabla_Z([\nabla_X, \nabla_Y]W)$$

$$= ([\nabla_X, [\nabla_Y, \nabla_Z]](W) + [\nabla_Y, [\nabla_Z, \nabla_X]](W) + [\nabla_Z, [\nabla_X, \nabla_Y]](W),$$

and the fact this vanishes follows from the Jacobi identity for commutators of linear operators.

An important corollary is the *contracted Bianchi identity*, which is useful in General Relativity. It states:

$$div(Ric) - \frac{1}{2}\nabla S = 0,$$

where Ric and S are the (1,1) Ricci tensor and the scalar curvature, and $div(Ric) = \sum_i \nabla_{e_i}(Ric)(e_i)$.

To see this, compute in an orthonormal frame which is geodesic at p (so $\nabla_{e_i} e_j(p) = 0$.) Then at p:

$$\langle div(Ric), X\rangle = \sum_i \langle (\nabla_{e_j} Ric)e_j, X\rangle = \sum_{i,j} \langle (\nabla_{e_j} R)(e_j, e_i)e_i, X\rangle = \sum_{i,j} \langle (\nabla_{e_j} R)(e_i, X)e_j, e_i\rangle$$

$$= \sum_{i,j} [-(\nabla_{e_i} R)(X, e_j)e_j, e_i\rangle - \langle (\nabla_X R)(e_j, e_i)e_j, e_i\rangle]$$

(from the differential Bianchi identity just proved)

$$= -\sum_j \langle (\nabla_{e_j} Ric)(X), e_j\rangle - X(S) = -\langle div(Ric), X\rangle - X(S),$$

proving the claim.

5. The Gauss and Codazzi equations for hypersurfaces.

Let $M \subset \bar{M}$ be a hypersurface (codimension 1-submanifold) with the Riemannian metric induced from $\bar{M}$. For vector fields $X \in \chi_M, Y \in \bar{\chi}_M$ (the space of vector fields on $\bar{M}$ restricted to M) we have the tangent-normal decomposition (with respect to a unit normal vector $N \in \bar{\chi}_M$):

$$\bar{\nabla}_X Y = \nabla_X Y + A(X,Y)N,$$

where A is the second fundamental form (recall $A(X,Y) = -\langle \bar{\nabla}_X N$. Iterating this formula, we find:

$$\bar{\nabla}_X \bar{\nabla}_Y Z = \bar{\nabla}_X(\nabla_Y Z + A(Y,Z)N) = \nabla_X \nabla_Y Z + X(A(Y,Z))N + A(Y,Z)\bar{\nabla}_X N.$$

skew-symmetrising and taking tangential components (using $[X,Y](p) = 0$, for a frame geodesic at p):

$$[(\bar{R}(X,Y)Z]^{tan} = R(X,Y)Z + A(Y,Z)(\bar{\nabla}_X N)^{tan} - A(X,Z)(\bar{\nabla}_Y N)^{tan}.$$

Taking inner product with $W \in \chi_M$, we find for the (4,0) Riemann tensors:

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + A(Y,Z)\langle \bar{\nabla}_X N, W\rangle - A(X,Z)\langle \bar{\nabla}_Y N, W\rangle$$
$$= \langle R(X,Y)Z, W\rangle - A(Y,Z)A(X,W) + A(X,Z)A(Y,W).$$

Using the Kulkarni-Nomizu product, we obtain for the (4,0) curvature tensor of M and $\bar{M}$ the relation:

$$\bar{Riem} = Riem + A \odot A.$$

This immediately implies, for the sectional curvatures along the 2-plane $span\{X,Y\}$:

$$\bar{\sigma}_{XY} = \sigma_{XY} - A(X,X)A(Y,Y) + A(X,Y)^2 \quad \{X,Y\} \text{ orthonormal}$$

This is the general *Gauss equation* for hypersurfaces in a Riemannian manifold. On the other hand, taking inner product of the first relation above with the unit normal N:

$$\langle \bar{R}(X,Y)Z, N\rangle = X(A(Y,Z)) - Y(A(X,Z)) = (\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z).$$

This is the *Codazzi equation*. In terms of the shape operator:

$$(\nabla_X S)Y - (\nabla_Y S)X = -\bar{R}(X,Y)N.$$

In particular, $(\nabla_X S)Y - (\nabla_Y S)X = 0$ if $\bar{M}$ is flat (or, more generally, of constant curvature.)