2. TOTALLY GEODESIC/ TOTALLY UMBILIC SUBMANIFOLDS.

Induced connection and second fundamental form. Let $\bar{M}$ be Riemannian or Lorentzian, $M \subset \bar{M}$ a submanifold. We assume the induced metric on M is nondegenerate, so M is also either Riemannian or Lorentzian. *Remark:* The null cone Λ is a degenerate hypersurface in $\mathbb{M}^{n+1}$ if $n \geq 2$.

Denote by $\bar{\chi}_M$ the space of vector fields on $\bar{M}$, restricted to M, and by χ_M the space of vector fields on M; let $\bar{\nabla}, \nabla$ be the Levi-Civita connections on $\bar{M}$, resp. M. For $X \in \chi_M, Y \in \bar{\chi}_M,$, $\bar{\nabla}_X Y \in \bar{\chi}_M$ is defined by taking local extensions $\bar{X}, \bar{Y}$ of X, Y to $\bar{M}$, then restricting $\bar{\nabla}_{\bar{X}} \bar{Y}$ to M (this is independent of the extensions). By uniqueness of the L-C connection, the orthogonal projection of $\bar{\nabla}_X Y$ onto TM equals $\nabla_X Y$; the normal component is a quadratic form on M with values in the *normal bundle* $NM \subset T\bar{M}$, the *second fundamental form* of M in $\bar{M}$. Thus we have the decomposition:

$$\bar{\nabla}_X Y = (\bar{\nabla}_X Y)^{tan} + (\bar{\nabla}_X Y)^{nor} = \nabla_X Y + II(X, Y).$$

In particular, if $\alpha(t)$ is a parametrized curve on M and $V(t) \in T_{\alpha(t)} M$ is a tangent vector field on M along α, we have for the covariant derivatives:

$$\frac{\bar{D}V}{dt} = \frac{DV}{dt} + II(\alpha', V),$$

and for the covariant acceleration vectors:

$$\frac{\bar{D}\alpha'}{dt} = \frac{DV}{dt} + II(\alpha', \alpha').$$

We see from this that α *is a geodesic on* M *if and only if* $\frac{\bar{D}\alpha'}{dt}$ *is normal to* M.

Example. Geodesics on hyperquadrics: spheres, hyperbolic spaces and de-Sitter spacetimes.

For the unit sphere $S^n \subset \mathbb{R}^{n+1}$, the great circle parametrized by arc length:

$$\gamma(t) = (\cos t)u + (sint)e, \quad e \cdot u = 0, |e| = 1, t \in \mathbb{R}$$

is the unique geodesic with $\gamma(0) = u \in S^n$, $\gamma'(0) = e \in T_u S^n = u^{\perp}$. Note $\gamma''(t) = -\gamma(t)$, so

$$II_u(v, v) = -u, \quad |v| = 1, |u| = 1.$$

S^n "bends" in the direction of the inward unit normal.

Now consider hyperbolic space, the upper component of the hyperquadric in Minkowski space:

$$H^n(1) = \{u \in \mathbb{M}^{n+1}; \langle u, u\rangle = -1, u_0 > 0\}; T_u H^n = u^\perp.$$

The parametrized curve in H^n:

$$\gamma(t) = (\cosh t)u + (\sinh t)e, \quad e \in u^\perp \text{ (spacelike) }, |e| = 1, t \in \mathbb{R}$$

has acceleration vector $\gamma''(t) = \gamma(t) \in N_{\gamma(t)}M$, hence is the unique geodesic (parametrized by arc length) with initial conditions (u, e).

De Sitter spacetime is the hyperquadric:

$$M^n = \{u \in \mathbb{M}^{n+1}; \langle u, u\rangle = 1\}(u \text{ spacelike }), T_u M = u^\perp \text{ (Lorentzian) }.$$

There are three cases for $e \in T_u M$:

(i) If e is timelike with $\langle e, e\rangle = -1$, $\gamma(t) = (\cosh t)u + (\sinh t)e, t \in \mathbb{R}$, satisfies $\gamma''(t) = \gamma(t)$, hence is the unique geodesic with initial conditions (u, e). Note γ' is timelike, with $|\gamma'| = 1$. (γ is a branch of a hyperbola in the 2-dimensional subspace of $\mathbb{M}^{n+1}$ spanned by u, e.

(ii) If e is *null*, then $\gamma(t) = u + te, t \in \mathbb{R}$ is a line contained in M, satisfying $\gamma'' \equiv 0$, so it is the geodesic with i.c. (u, e), and it is a null curve ($\alpha'(t)$ is null).

(iii) If e is spacelike, $|e| = 1$, then the circle parametrized by arc length $\gamma(t) = (\cos t)u + (\sin t)e$ is a curve on M satisfying $\gamma''(t) = -\gamma(t)$ (a normal vector); so γ is the (spacelike) geodesic with initial conditions (u, e).

Completeness and geodesic connectedness. A Riemannian (or Lorentzian) manifold is *geodesically complete* if every inextendible geodesic is defined over $\mathbb{R}$. As seen above, all hyperquadrics are (geodesically) complete. (Note also that each geodesic lies entirely in the two-dimensional subspace of $\mathbb{M}^{n+1}$ spanned by its initial conditions.

In the Riemannian case, the *Hopf-Rinow theorem* guarantees that a geodesically complete manifold is *geodesically connected*: any two points can be joined by at least one minimizing geodesic. This is not true in the Lorentzian case; for example, de Sitter spacetime is not geodesically connected.

Indeed let $u, v \in S^n(1) \subset \mathbb{M}^{n+1}$ be non-antipodal points. Then u, v span a two-dimensional subspace Π of $\mathbb{M}^{n+1}$. Suppose there is a geodesic γ of

$S^n(1)$ from u to v; then this geodesic to lie in this subspace. There are three cases:

(i) Π is spacelike. Then γ is an arc of the spacelike circle $\Pi \cap S^n(1)$, and $|\langle u, \gamma(t)\rangle| < 1$ for all $t \notin 2\pi\mathbb{Z}$; in particular $|\langle u, v\rangle| < 1$.

(ii)Π is degenerate. Let $\{u, e\}$ be an orthogonal basis of Π, with $|u| = 1$ and e null. Then $\Pi \cap S^n_1$ contains the the two parallel lines $\pm u + te, t \in \mathbb{R}$; and it is not hard to show it *equals* the union of these lines. Hence u, v are connected if on the same line (which happens iff $\langle u, v\rangle = 1$), not connected if on different lines (when $\langle u, v\rangle = -1$.

(iii)Π is Lorentzian. Then letting $\{u, w\}$ be an orthogonal basis of Π with $u \in S^n_1$ (hence spacelike) and w timelike, $|w| = 1$, the intersection $\Pi \cap S^n_1$ is the hyperbola:

$$\{xu + yw; x, y \in \mathbb{R}, x^2 - y^2 = 1\}.$$

Clearly if u, v are connected they lie on the same arc of the hyperbola, that is, if $v = xu + yw$ with $x > 1$; this happens iff $\langle u, v\rangle = x > 1$.

We conclude that if u, v are non-antipodal points connected by a geodesic, then $< \langle u, v\rangle > -1$. So if $\langle u, v\rangle \leq -1$ they're not connected. This can certainly happen for u, v in S^n_1: just let $u = (\cosh t)e_1 + (\sinh t)e_0$, $v = (-\cosh t)e_1 + (\sinh t)e_0$, for any $t \in \mathbb{R}$.

Conversely, it is easy to see from the above that if u, v are not antipodal and $\langle u, v\rangle > -1$, they're connected by a geodesic: a spacelike circular arc (non-unique) if $|\langle u, v\rangle| < 1$, a null line segment if $\langle u, v\rangle = 1$, a timelike arc of hyperbola if $\langle u, v\rangle > 1$.

Totally geodesic submanifolds.

Proposition 3. Let $M \subset \bar{M}$ be a non-degenerate submanifold ($\bar{M}$ Riemannian or Lorentzian). The following are equivalent:

(1) The second fundamental form of M in $\bar{M}$ vanishes identically.

(2) Any geodesic of M is also geodesic in $\bar{M}$.

(3) If $v \in T_u\bar{M}$ is in fact in T_uM, the $\bar{M}$ geodesic arc γ with initial condition (u, v) lies in M.

(4) If α is a curve on M and $V(t)$ is tangent to M and ∇-parallel along α, then it is also $\bar{\nabla}$-parallel.

Proof. Exercise. Prove (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (1). For (2) $\Rightarrow$ (1) use a polarization identity. Then prove (3) $\Rightarrow$ (1) (using the decomposition of the $\bar{\nabla}$-acceleration vector of a curve on M) and (2) $\Rightarrow$ (3) (the geodesic on M

with the same i.c. as γ is also geodesic on $\bar{M}$, hence coincides with γ, by uniqueness.

Definition. If any of these conditions is satisfied, we say M is a *totally geodesic submanifold* of $\bar{M}$.

Problem. (i) Show that the intersection of any subspace of $\mathbb{R}^{n+1}$ with the unit sphere S^n is a totally geodesic submanifold of S^n. (ii) Show that the intersection of any (necessarily timelike) subspace of $\mathbb{M}^{n+1}$ with $H^n(1)$ is a totally geodesic submanifold of hyperbolic space. (iii) Formulate and prove a similar statement for the intersections of subspaces of $\mathbb{M}^{n+1}$ with deSitter spacetime $S^n_1(1)$ (note there are three cases to consider). *Hint:* use condition (3) in Proposition 3 and the earlier description of geodesics in hyperquadrics.

Problem. Show that the totally geodesic submanifolds of $\mathbb{R}^{n+1}$ and $\mathbb{M}^{n+1}$ are the affine subspaces.

Totally umbilic hypersurfaces. If M has codimension 1 in $\bar{M}$, at least locally M admits a "unit" normal vector field $U \in T\bar{M}$ (Possibly $\langle U, U\rangle = -1$). In this case we have a scalar second fundamental form $A(X,Y)$ and the decomposition (for $X \in \chi_M, Y \in \bar{\chi}_M$):

$$\bar{\nabla}_X Y = \nabla_X Y + \epsilon_U A(X,Y)U,$$

with $A(X,Y) = \langle II(X,Y), U\rangle$ and $\epsilon_U = \langle U, U\rangle = \pm 1$.

Definition. We say M is *totally umbilic* at $u \in M$ if $II_p(X,Y) = \langle X, Y\rangle z$, for some vector $z \in (T_u M)^\perp$, the *normal curvature vector* at u (not necessarily a unit vector). M is a totally umbilic hypersurface it it is totally umbilic at each point. Equivalently, at each $u \in M$ the *shape operator* (the self-adjoint operator associated to the quadratic form A by the metric) is a multiple of the identity on $T_u M$.

Using $S(X) = -\bar{\nabla}_X U$, it is easy to check that the unit sphere in $\mathbb{R}^{n+1}$, as well as hyperbolic space and deSitter spacetime (as hyperquadrics in in $\mathbb{M}^{n+1}$), are totally umbilic hypersurfaces.

From the definition of $II(X,Y)$, we have:

$$\langle II(X,Y), U\rangle = \langle \bar{\nabla}_X Y, U\rangle = -\langle \bar{\nabla}_X U, Y\rangle = \langle S(X), Y\rangle$$

$$II_u(X,Y) = \langle S(X), Y\rangle U(u) \text{ (U spacelike) } ; II_u(X,Y) = -\langle S(X), Y\rangle \text{ (U timelike) } .$$

For S^n we let $U(u) = -u$, so $S(X) = X$ for $X \in T_u S^n$. Thus $II_u(X,Y) = -\langle X, Y\rangle u$, and the normal curvature vector is $z(u) = -u$. (Starting with the outward normal $U(u) = u$ would lead to the same result–the sphere "bends towards" its normal curvature vector.)

For $H^n \subset \mathbb{M}^{n+1}$, we let $U(u) = u$ (timelike), so $S(X) = -X$. Then $II_u(X, Y) = \langle X, Y \rangle U$, and the normal curvature vector is $z(u) = u$; the hypersurface "bends towards" its normal curvature vector.

For $S_1^n \subset \mathbb{M}^{n+1}$, the deSitter spacetime, again taking $U(u) = u$ as the unit normal at u (spacelike) we find $S(X) = -X, II_u(X, Y) = -\langle X, Y \rangle U$, so the normal curvature vector is $z(u) = -u$. The hypersurface "bends towards" its normal curvature vector along spacelike curves, and away from it along timelike directions.