FIRST AND SECOND VARIATIONS OF AREA FOR HYPERSURFACES

\((\bar{M}, g)\) is a Riemannian manifold, \(M\) a compact submanifold with boundary, with the induced metric and Riemannian connection. We consider a variation of \(M\) in \(\bar{M}\), with fixed boundary:

\[
f : M \times I \to \bar{M}, \quad f_0 = id_M, f_{|\partial M} = id_{\partial M}.
\]

We assume the \(f_t : M \to \bar{M}\) are embeddings, and let \(M_t = f_t(M)\). Let \(\omega_t\) be the induced volume form on \(M_t\). Then:

\[
vol(M_t) = \int_M f_t^* \omega_t, \quad vol(M) = \int_M \omega_0.
\]

The associated variational vector field is:

\[
V = \partial_t f \in T\bar{M}|_{M_t}.
\]

(Note \(V_{|\partial M} \equiv 0\).) Recall the definition of second fundamental form \(B(X,Y)\) with values in the normal bundle \(NM\) of \(M\) in \(\bar{M}\): given \(X,Y\) vector fields on \(M\), we have the decomposition:

\[
\bar{\nabla}_X Y = (\bar{\nabla}_X Y)^TM + (\bar{\nabla}_X Y)^NM = \nabla_X Y + B(X,Y).
\]

The mean curvature vector \(\vec{H}\) is defined as the trace of \(B\):

\[
\vec{H}(p) = \sum_i (\bar{\nabla}_{e_i} e_i)^{NM}, \quad (e_i) \text{ o.n. at } p.
\]

Remark. If \(M\) has codimension 1 in \(\bar{M}\) and \(N\) is a (local) choice of unit normal, we have:

\[
\vec{H} = HN, \quad B(X,Y) = A(X,Y)N, \quad H = \sum_i A(e_i, e_i),
\]

where \(A\) and \(H\) are the scalar second fundamental form and mean curvature. Note that although \(A\) and \(H\) depend on the choice of \(N\), \(B\) and \(\vec{H}\) do not.

Theorem. (First variation of area.)

\[
\frac{d}{dt} vol(M_t)|_{t=0} = - \int_M \langle \vec{H}, V \rangle \omega_0.
\]
Recall the pullback of the volume form $\omega_t$ of $M_t$ under $f_t$ is defined by:

$$f_t^*\omega_{t\mid p}(e_1, \ldots, e_n) = \omega_{t\mid f_t p}(df_t e_1, \ldots, df_t e_n) = \det(df_t)\omega_t(e_1, \ldots, e_n)\mid_{f_t p} = \det(df_t).$$

(Here we assume the positive orthonormal frame $(e_i)$ of $TM$ has been extended to orthonormal vector fields in a neighborhood of $p$ in $M$, tangent to the $M_t$.)

We need a standard lemma expressing $\det df_t$ in terms of the metric.

**Lemma 1.** In a $n$-dimensional vector space $E$ with inner product and an orientation let $\{v_1, \ldots, v_n\}$ be a positive basis. Then:

$$\det[v_1| \ldots |v_n] = \sqrt{\det((v_i, v_j))}.$$ 

Let $(e_i)$ be a positive orthonormal basis of $E$. The proof follows from noting the matrix on the left has entries $A_{ij} = \langle v_j, e_i \rangle$, while the matrix $P_{ij} = \langle v_i, v_j \rangle$ is easily seen to equal $A^T A$:

$$P_{ij} = \langle v_i, v_j \rangle = \sum_k \langle v_i, e_k \rangle \langle e_k, v_j \rangle = \sum_k A_{ki} A_{kj} = (A^T A)_{ij}.$$ 

Thus $\det P = (\det A)^2$. Applying this with:

$$g^t_{ij} = \langle df_t e_i, df_t e_j \rangle, g(t) = \det(g^t_{ij}), df_t = [df_t e_1| \ldots |df_t e_n],$$

we find: $\det(df_t) = \sqrt{g(t)}$. Thus:

$$\frac{d}{dt} \text{vol}(M_t)\mid_{t=0} = \int_M \frac{d}{dt} \det(df_t)\mid_{t=0} \omega_0 = \int_M \frac{1}{2} \left( \frac{dg}{dt} \right)\mid_{t=0} \omega_0.$$ 

**Lemma 2.** Let $A(t)$ be a differentiable family of invertible linear maps of $E^n$, such that $A(0) = I$. Then:

$$\frac{d}{dt}(\det A(t))\mid_{t=0} = \text{tr} \dot{A}(0).$$

**Proof.** $\det A(t) = \omega(A(t)e_1, \ldots, A(t)e_n)$. With $\dot{A}(0)e_i = \sum_j a^j_i e_j$:

$$\frac{d}{dt}(\det A(t)) = \sum_i \omega(e_1, \ldots, \dot{A}(0)e_i, \ldots, e_n) = \sum_i a^j_i = \text{tr} \dot{A}(0).$$
Exercise 1. Without the hypothesis $A(0) = I$, show that:

$$(\det A(t))^{-1} \frac{d}{dt} \det A(t)|_{t=0} = tr(\dot{A}(0)A^{-1}(0)).$$

Proof of theorem. Fix $p \in M$ and a coordinate system in a neighborhood of $p$ in $M \times I$ with coordinate vector fields $\partial_t, \partial_i$, where $\partial_t|_p = e_i|_p$ and the vector fields $e_i$ are orthonormal, tangent to the submanifolds $M_t$ in a neighborhood of $p$ in $M$, and geodesic at $p$: $\nabla_{e_i} e_j(p) = 0$. Let $e^t_i = df \cdot e_i$.

Thus also: $[V, e^t_i] = [df \cdot \partial_t, df \cdot e_i] = df[\partial_t, e_i] = 0$ at $p$.

This implies (at $p$):

$$\frac{1}{2} \partial_t g^t_{ii}|_{t=0} = \langle \nabla_V e^t_i, e^t_i \rangle|_{t=0} = \langle \nabla e^t_i V, e^t_i \rangle|_{t=0} = e_i(\langle V, e_i \rangle) - \langle V, \nabla e^t_i e^t_i \rangle = e_i(\langle V, e_i \rangle) - \langle V, B(e_i, e_i) \rangle - \langle V, \nabla e_i e_i \rangle,$$

where the last term vanishes at $p$. Adding over $i$, we find at $t = 0$:

$$\frac{1}{2} \frac{dg}{dt} = \frac{1}{2} \sum_i \partial_t g^t_{ii} = -\langle V, \vec{H} \rangle + \sum_i e_i(\langle V, e_i \rangle).$$

We conclude:

$$\frac{d}{dt} \omega|_{t=0} = -\langle V, \vec{H} \rangle \omega_0 + \sum_i e_i(\langle V, e_i \rangle) \omega_0.$$ 

Lemma 3. Let $\alpha = *(V^* ) \in \Omega^{n-1}$, where $V^* \in \Omega^1$ is the 1-form dual to $V$. Then $\sum_i e_i(\langle V, e_i \rangle) \omega_0 = d\alpha$.

Assuming the lemma, integrating $\frac{d}{dt} \omega = -\langle V, \vec{H} \rangle \omega_0 + d\alpha$ and using Stokes’ theorem and $\alpha = 0$ on $\partial M$, we conclude the proof of the theorem:

$$\frac{d}{dt} \text{vol}(M_t)|_{t=0} = \int_M \frac{d}{dt} \omega|_{t=0} = -\int_M -\langle \vec{H}, V \rangle \omega_0 + \int_{\partial M} \alpha = -\int_M \langle \vec{H}, V \rangle \omega_0.$$ 

Proof of lemma 3. Recall the expression for exterior derivative:

$$d\alpha(e_1, \ldots, e_n) = \sum_{i=1}^n e_i(\alpha(e_1, \ldots, \hat{e}_i, \ldots, e_n)) - \sum_{i<j} (-1)^{i+j} \alpha([e_i, e_j], \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n),$$

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Since we’re assuming \( \nabla_{e_i e_j}(p) = 0 \), the term involving the Lie brackets \([e_i, e_j]\) vanishes at \( p \). On the other hand, since \( V^* = \sum_k (V, e_k) \theta_k \) (where \( \theta_k \in \Omega^1_M \) is dual to \( e_k \)):
\[
\alpha = \ast(V^*) = \sum_k (-1)^{k-1} (V, e_k) \theta_1 \wedge \ldots \wedge \theta_k \wedge \ldots \wedge \theta_n \Rightarrow \alpha(e_1, \ldots, \hat{e}_i, \ldots, e_n) = (-1)^{i-1} (V, e_i).
\]
This clearly implies:
\[
d\alpha(e_1, \ldots, e_n)|_p = \sum e_i \langle (V, e_i) \rangle \omega_0(e_1, \ldots, e_n),
\]
as claimed.

**Remark.** Note that if the variation is normal, that is, \( \langle V, e_i \rangle = 0 \) for all \( i \), it follows that \( \alpha = 0 \) on \( \partial M \), so the result is true for all normal variations, even without the boundary condition \( f|_{\partial M} = id_{\partial M} \).

**The second variation formula.** We consider only normal variations of a minimal surface \( M \):
\[
H = 0, \quad \partial_t f = V = uN,
\]
where \( u \) is a function on \( M \). The goal is to show that:
\[
\frac{d^2}{dt^2} \text{vol}(M_t)|_{t=0} = - \int_M (\Delta_M u + |A|^2 u - 2\text{Ric}_N u) u \omega_0,
\]
where \( \Delta_M \) is the Laplace-Beltrami operator and \( 2\text{Ric}_N \) the Ricci curvature of \( M \) in the direction \( N \).

From the first variation formula:
\[
\frac{d^2}{dt^2} \text{vol}(M_t) = - \int_M \frac{d}{dt} \langle V, \vec{H} \rangle \omega_t - \int_M \langle V, \vec{H} \rangle \frac{d\omega_t}{dt},
\]
which at \( t = 0 \) equals \( - \int_M \partial_t \langle V, \vec{H} \rangle |_{t=0} \omega_0 \), since we assume \( \vec{H} = 0 \) on \( M \).

Now fix \( p \in M \) and a local frame \( (e_i) \) in a neighborhood of \( p \) in \( M \), orthonormal on \( M \) and geodesic at \( p \): \( \nabla_{e_i} e_j(p) = 0 \). We assume the \( (e_i) \) are transported to \( M_t \) by \( df_t \), and set \( g_{ij} = \langle e_i, e_j \rangle \). In particular, \( [V, e_i] = 0 \).

From the definition of \( \vec{H} \):
\[
\frac{d}{dt} \langle V, \vec{H} \rangle |_{t=0} = \sum g^{ij} \frac{d}{dt} |_{t=0} \langle \nabla_{e_i} e_j, V \rangle + \sum g^{ij} \frac{d}{dt} |_{t=0} \langle \nabla_{e_i} e_j, V \rangle |_{t=0} := I + II.
\]
Noting that \( g_{ij} = \delta_{ij} \) at \( t = 0 \), we have, at \( t = 0 \) and at \( p \):

\[
\frac{dg_{ij}^t}{dt} = - \frac{dg_{ij}}{dt} = -V((e_i, e_j)) = -\langle \bar{\nabla} V e_i, e_j \rangle - \langle e_i, \bar{\nabla} V e_j \rangle = -\langle \nabla e_i V, e_j \rangle - \langle e_i, \nabla e_j V \rangle = 2\langle B(e_i, e_j), V \rangle,
\]

using \( \nabla e_i e_j(p) = 0 \). Thus:

\[
I = 2 \sum_{i,j} \langle B(e_i, e_j), V \rangle^2 = 2u^2|A|^2,
\]

for normal variations in the codimension 1 case.

For term \( II = \sum_j \langle \bar{\nabla} V \nabla e_j, V \rangle \), using \([V, e_i] = 0\):

\[
\langle \bar{\nabla} V \nabla e_j, V \rangle = \langle \nabla e_j \bar{\nabla} V e_j, V \rangle + \langle \bar{R}(e_j, V)e_j, V \rangle = e_j(\langle \bar{\nabla} V e_j, V \rangle - |\nabla e_j V|^2) - \langle \bar{R}(e_j, V)e_j \rangle = e_j(\langle \nabla e_j V, V \rangle - |\nabla e_j V|^2 - \langle \bar{R}(e_j, V)e_j \rangle).
\]

In the codimension 1 case, with \( V = uN \):

\[
|\bar{\nabla} e_j V|^2 = |e_j(u) + u\bar{\nabla} e_j N|^2 = e_j(u)^2 + u^2|\nabla e_j N|^2,
\]

adding over \( j \):

\[
\sum_j |\bar{\nabla} e_j V|^2 = |\nabla u|^2 + u^2 \sum_{i,j} \langle \nabla e_j N, e_i \rangle^2 = |\nabla u|^2 + u^2 \sum_{i,j} A(e_i, e_j)^2 = |\nabla u|^2 + u^2|A|^2,
\]

while:

\[
e_j(\langle \bar{\nabla} e_j V, V \rangle) = e_j(\langle e_j(u)N + u\bar{\nabla} e_j N, uN \rangle) = e_j(e_j(u)u) = e_j(e_j u) + u^2 e_j(u)^2,
\]

so:

\[
\sum_j e_j(\langle \nabla e_j V, V \rangle) = \sum_j u(e_j(e_j u) - (\bar{\nabla} e_j e_j)u) + |\nabla u|^2 = u \Delta M u + |\nabla u|^2.
\]

Combining the above, we find:

\[
\sum_j \langle \bar{\nabla} V \bar{\nabla} e_j, V \rangle = u \Delta M u - u^2 \overline{Ric}_N - u^2 |A|^2.
\]

Thus:

\[
I + II = u \Delta M u + u^2 |A|^2 - u^2 \overline{Ric}_N,
\]

as we wished to show, since:

\[
\frac{d^2}{dt^2} \ vol(M_t) = - \int_M (I + II) \omega_0.
\]