NOTES ON P.BÉRARD'S SURVEY

Main theorem. Let (M^n, g) be a compact Riemannian manifold without boundary, satisfying the Ricci curvature lower bound:

$$ric_{min}(M)\mathrm{diam}(M)^2 \geq -(n-1)\alpha^2.$$

Let E be a rank l Riemannian vector bundle over M, with Riemannian connection ∇. Suppose Δ_H is a second-order elliptic differential operator on smooth sections s of E, satisfying the pointwise Bochner-type identity:

$$\Delta_H s = \nabla^*\nabla s + \mathcal{R}s,$$

where $\nabla^*\nabla s = -\sum_i \nabla^2_{e_i,e_i} s$ is the 'connection Laplacian' of ∇ ((e_i) an arbitrary local orthonormal frame) and $\mathcal{R}$ a zero-order operator depending on the curvature tensor of g and on ∇. Denote by $\mathcal{H}$ the kernel of Δ_H.

Then there exists a positive number $A(n, \alpha)$ so that:

$$\mathcal{R}_{min}\mathrm{diam}(M)^2 > -\Lambda^2 \text{ with } 0 < \Lambda < A(n, \alpha) \Rightarrow \dim(\mathcal{H}) \leq l.$$

Here $\mathcal{R}_{min} = \min\{\langle \mathcal{R}_p s, s\rangle; p \in M, |s|(p) = 1.\}$

More precisely, there exists a constant $b = b(n, \alpha, \Lambda)$ such that $\dim(\mathcal{H}) \leq bl$; and $b \to 1$ when $\Lambda \to 0_+$.

Remark: For instance, if Δ_H is the Hodge Laplacian on p-forms, this says we can get universal bounds on betti numbers, even when the curvature operator is allowed to be a little bit negative. (cp. Gromoll-Meyer's theorem.) It's a kind of 'stability result' for the Bochner method.

1. Isoperimetric profile and the Sobolev constant. For $q > 1$ and $1 < q < p$ such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, define:

$$S_{p,q}(M) = \sup\{\frac{||f||_p}{||df||_q}; f \in W^{1,q}(M), f \not\equiv 0, \int_M f = 0\}.$$

We have the Sobolev inequality corresponding to the embedding $W^{1,q} \hookrightarrow L^p$:

$$||f||_p \leq S_{p,q}(M)||df||_q + vol(M)^{-1/n}||f||_q.$$

Theorem 1. (i) Suppose we have the inequality of isoperimetric profiles:

$$h_M(s) \geq h_{S^n_R}(s), \quad s \in [0, 1],$$

for some $R > 0$. Then:

$$S_{p,q}(M) \leq (\frac{vol(M)}{vol(S^n_R)})^{-1/n} S_{p,q}(S^n_R).$$

(This is proved by rearrangement of f to a function f^* on S^n_R.)

(ii) $S_{p,q}(S^n_R) = S_{p,q}(S^n)$, a.k.a. 'Sobolev quotients at the critical exponent are dilation-invariant' (Proved by a scaling argument- easy exercise.)

(iii) It follows from (i) and (ii) that:

$$S_{p,q}(M) \leq (\frac{vol(M)}{vol(S^n_R)})^{-1/n} S_{p,q}(S^n) = vol(M)^{-1/n} R S_{p,q}(S^n) vol(S^n)^{1/n}.$$

(iv) In particular, under the assumption in (i) on isoperimetric profiles, the Sobolev inequality for $q = 2, p = \frac{2n}{n-2}$ reads:

$$||f||_{\frac{2n}{n-2}} \leq vol(M)^{-1/n}[R\sigma_n ||df||_2 + ||f||_2], \quad \sigma_n = S_{\frac{2n}{n-2},2}(S^n) vol(S^n)^{1/n}.$$

2. Ricci lower bound controls the isoperimetric profile.

Theorem 2. [Bérard-Besson-Gallot, Inventiones 1985] Suppose we have the Ricci lower bound:

$$ric_{min} diam(M)^2 \geq -(n-1)\alpha^2.$$

Then there exists a positive constant $a(n, \alpha)$ so that:

$$diam(M) \frac{h_M(s)}{h_{S^n}(s)} \geq a(n, \alpha).$$

Equivalently, with $R = \frac{diam(M)}{a(n,\alpha)}$, we have:

$$h_M(s) \geq h_{S^n_R}(s) = R^{-1} h_{S^n}(s).$$

3. Kato's inequalities. $E \to M$ Riemannnian vector bundle, with Riemannian connection ∇. $s \in \Gamma(E)$ smooth section. Assume M compact (for simplicity). Although $|s|^2$ is a smooth function on M, in general $|s|$ is not smooth (since s may have zeros), only locally Lipschitz (in particular differentiable a.e.), since:

$$||s|(x) - |s|(y)| \leq |s(x) - s(y)|.$$

First Kato inequality. The distributional derivative $d|s|$ is in $L^2(M)$, and satisfies, pointwise a.e.:

$$|d|s|| \leq |\nabla s|.$$

Proof. Consider, for $\epsilon > 0$, the smooth function $f_\epsilon = (|s|^2 + \epsilon)^{1/2}$. Let (e_i) be a local o.n. frame. We have:

$$e_i(f_\epsilon) = \frac{\langle \nabla_{e_i} s, s \rangle}{(|s|^2 + \epsilon)^{1/2}} \leq \frac{|\nabla_{e_i} s||s|}{(|s|^2 + \epsilon)^{1/2}} \leq |\nabla_{e_i} s|.$$

Adding over i, we conclude: $|df_\epsilon| \leq |\nabla s|$, pointwise on M.

Let $d|s|$ be the distributional derivative of $|s|$, and let $\alpha \in \Omega^1_M$ be a 'test 1-form' (smooth, with compact support.) Then, as a linear functional,

$$(d|s|)[\alpha] := \int_M \langle \delta\alpha, |s| \rangle = \lim_\epsilon \int_M (\delta\alpha) f_\epsilon = \lim_\epsilon \int_M \langle \alpha, df_\epsilon \rangle.$$

Thus:

$$|(d|s|)[\alpha]| \le \lim_\epsilon \int_M |\alpha||df_\epsilon| \le \int_M |\alpha||df_\epsilon| \le \int_M |\alpha||\nabla s| \le ||\alpha||_{L^2}||\nabla s||_{L^2}.$$

Thus in fact $d|s|$ is defined in $L^2(M)$, and satisfies the pointwise a.e. bound $|d|s|| \le |\nabla s|$.

Consider now the distribution $\Delta|s|$, which is in the L^2 Sobolev space $H^{-1}(M)$, the dual of $H^1(M)$. We have:

Second Kato inequality. $|s|\Delta|s| \ge -\langle \nabla^*\nabla s, s \rangle$,

as distributions (this means $|s|(\Delta|s|)[\phi] \ge \phi\langle \nabla^*\nabla s, s \rangle$, for any smooth nonnegative test function $\phi \ge 0$).

Proof. (i) It is easy to show that $d|s|^2 = 2|s|d|s|$ as distributions; that is, for any smooth test 1-form α:

$$\int_M |s|^2 \delta\alpha = 2\int_M |s|\langle d|s|, \alpha \rangle.$$

(We already know $d|s|$ is in $L^2\Omega^1_M$, so the integral is defined.) First, for any smooth 1-form α:

$$\delta(\alpha|s|) = |s|\delta\alpha - \langle \alpha, d|s| \rangle \quad \text{as distributions.}$$

This implies that, for smooth 1-forms α:

$$\int_M \langle d|s|, |s|\alpha \rangle = \int_M |s|\delta(|s|\alpha) = \int_M |s|^2\delta\alpha - \int_M \langle |s|\alpha, d|s| \rangle,$$

or:

$$\int_M |s|^2\delta\alpha = 2\int_M \langle |s|\alpha, d|s| \rangle = \int_M \langle \alpha, 2|s|d|s| \rangle,$$

as claimed.

(ii) We have: $2|s|\Delta|s| + 2|d|s||^2 = \Delta|s|^2$ (in the sense of distributions for $\Delta|s|$).

Proof: Let ϕ be a smooth test function (with compact support.) We may pair the H^1 function $2\phi|s|$ with the H^{-1} distribution $\Delta|s|$, and by definition the pairing is:

$$(\Delta|s|)[2\phi|s|] := -\int_M \langle d(2\phi|s|), d|s| \rangle = -2\int_M \phi|d|s||^2 - 2\int_M |s|\langle d\phi, d|s| \rangle.$$

Hence, using (i):

$$2(\Delta|s|)[\phi|s|]+2\int_M \phi|d|s||^2 = -2\int_M |s|\langle d\phi, d|s|\rangle = -\int_M \langle d\phi, d|s|^2\rangle = \int_M \phi(\Delta|s|^2),$$

as claimed.

(iii) We have the pointwise equality of smooth functions:

$$\Delta|s|^2 = -2\langle\nabla^*\nabla s, s\rangle + 2|\nabla s|^2.$$

Combining this with (ii) we have the equality (in the sense of distributions):

$$-\langle\nabla^*\nabla s, s\rangle + |\nabla s|^2 = |s|\Delta|s| + |d|s||^2.$$

And now use the first Kato inequality to estimate:

$$|s|\Delta|s| = -\langle\nabla^*\nabla s, s\rangle + |\nabla s|^2 - |d|s||^2 \geq -\langle\nabla^*\nabla s, s\rangle,$$

(as distributions), as claimed.

Lemma 3. In the setting of the main theorem, suppose the curvature operator in the Weitzenböck formula admits the lower bound:

$$\mathcal{R}_{min} \geq -\lambda^2.$$

Then if $s \in \mathcal{H}$ (i.e., $\Delta_H s = 0$), the Weitzenböck formula and Kato's second inequality imply:

$$|s|\Delta|s| \geq -\langle\nabla^*\nabla s, s\rangle = \langle\mathcal{R}s, s\rangle \geq -\lambda^2|s|^2,$$

in the distribution sense. We claim this implies $|s|$ satisfies the differential inequality (also in the sense of distributions):

$$-\Delta|s| \leq \lambda^2|s|.$$

Proof. We need to show that for any smooth function $\psi \geq 0$, we have:

$$\int_M \langle d|s|, d\psi\rangle \leq \lambda^2 \int_M |s|\psi.$$

The distributional inequality $|s|\Delta|s| \geq -\lambda^2|s|^2$ means that, for any smooth $\phi \geq 0$, we have:

$$\int_M \langle d\phi, |s|d|s|\rangle + \int_M \phi|d|s||^2 = \int_M \langle d(\phi|s|), d|s|\rangle \leq \lambda^2 \int |s|^2\phi.$$

With $f_\epsilon = \sqrt{|s|^2+\epsilon}$ as before, let $\phi = \frac{\psi}{f_\epsilon}$. Using:

$$d\phi = \frac{d\psi}{f_\epsilon} - \frac{\psi df_\epsilon}{f_\epsilon^2}, \quad df_\epsilon = \frac{|s|d|s|}{f_\epsilon}$$

we find:

$$\langle d\phi, |s|d|s|\rangle + \phi|d|s||^2 = \langle d\psi, \frac{|s|}{f_\epsilon} d|s|\rangle - \frac{\psi}{f_\epsilon^3}|s|^2|d|s||^2 + \frac{\psi}{f_\epsilon}|d|s||^2,$$

and this converges boundedly a.e. (as $\epsilon \to 0$) to $\langle d\psi, d|s|\rangle$. In addition,

$$|s|^2\phi = \frac{|s|^2\psi}{f_\epsilon} \to |s|\psi,$$

also boundedly a.e. We conclude:

$$\int_M \langle d|s|, d\psi\rangle \leq \lambda^2 \int_M |s|\psi,$$

as claimed.

4. Moser iteration. It's a classical PDE result that $W^{1,2}$ weak solutions of second-order linear elliptic equations satisfy L^∞ bounds in terms of their L^2 norms. The following theorem records the dependence of these bounds on Sobolev embedding constants.

Theorem 4. Let $f \geq 0$ be continuous and in $W^{1,2}(M)$, and satisfy the elliptic inequality $-\Delta f \leq af$ (in the sense of distributions), where $a \geq 0$ is a constant. Then f satisfies:

$$||f||_\infty^2 \leq \frac{B_n(x)}{V}||f||_2^2,$$

where $V = vol(M)$, $x = \gamma V^{1/n}\sqrt{a}$, $B_n(x) = \Pi_{i=0}^\infty(1 + \frac{xp^i}{\sqrt{2p^i-1}})^{2/p^i}$ with $p = \frac{n}{n-2}$ ($n \geq 3$) and γ the constant in the Sobolev inequality:

$$||f||_{2p} \leq \gamma||df||_2 + V^{-1/n}||f||_2.$$

Remark: It's a good Calculus exercise to verify that the infinite product defining $B_n(x)$ is indeed convergent.

Proof. The distributional inequality $-\Delta f \leq af$ yields (with f as test function, after approximation by smooth positive functions): $||df||_2 \leq \sqrt{a}||f||_2$; and using f^{2k-1} as test function ($k \geq 1$ not necessarily an integer):

$$||df^k||_2 \leq \sqrt{a}\frac{k}{\sqrt{2k-1}}||f||_{2k}^k.$$

Using the Sobolev inequality for f^k, we find:

$$||f^k||_{2p} \leq \gamma||df||_2 + V^{-1/n}||f^k||_2$$

$$\leq \gamma\sqrt{a}\frac{k}{\sqrt{2k-1}}||f||_{2k}^k + V^{-1/n}||f^k||_2$$

$$= (\gamma\sqrt{a}\frac{k}{\sqrt{2k-1}} + V^{-1/n})||f||_{2k}^k = (\frac{xk}{\sqrt{2k-1}} + 1)V^{-1/n}||f||_{2k}^k,$$

using $||f||_{2k}^k = ||f^k||_2$. Note also that $||f||_{2kp} = ||f^k||_{2p}^{1/k}$, so:

$$||f||_{2kp} \leq (1 + \frac{xk}{\sqrt{2k-1}})^{1/k} V^{-\frac{1}{nk}} ||f||_{2k} := z_k V^{-\frac{1}{nk}} ||f||_{2k}.$$

Since $p > 1$, this is a gain of integrability (with ratio p from the right-hand side to the left), and so we may iterate this estimate for $k = 1, p, p^2, \ldots$, obtaining, successively:

$$||f||_{2p} \leq z_1 V^{-1/n} ||f||_2$$

$$||f||_{2p^2} \leq z_p V^{-1/np} ||f||_{2p}$$

$$||f||_{2p^3} \leq z_{p^2} V^{-1/np^2} ||f||_{2p^2}$$

and so on; taking the infinite product, and recalling $||f||_\infty = \lim_{q\to\infty} ||f||_q$, we find:

$$||f||_\infty \leq (\Pi_{i=0}^\infty z_{p^i}) V^{-1/2} ||f||_2 = B_n(x)^{1/2} V^{-1/2} ||f||_2,$$

where we also used the elementary fact:

$$\frac{1}{n}(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots) = \frac{1}{n}\frac{1}{1-\frac{1}{p}} = \frac{p}{n(p-1)} = \frac{1}{2}$$

(recall $p = \frac{n}{n-2}$).

Remark: It is easy to show that $\lim_{x\to 0_+} B_n(x) = 1$, and in fact

$$B_n(x) \leq B_n(1)x^n \text{ for } x \geq 1.$$

5. Estimating dimension in terms of the ratio of norms L^∞/L^2.

Given a finite-dimensional subspace $F \subset C^\infty(E)$ of smooth sections, let $\{e_1, \ldots, e_N\}$ be a basis for F, orthonormal in the L^2 sense:

$$\langle e_i, e_j\rangle_{L^2} = \int_M \langle e_i(x), e_j(x)\rangle_{E_x} d\mu_M(x) = \delta_{ij}.$$

We claim that the function $f(x) = \sum_{i=1}^N |e_i(x)|^2$ is independent of the choice of basis. Indeed, since F is finite-dimensional, any other basis (f_i) of F satisfies:

$$f_i = \sum_j a_{ij} e_j, \quad i = 1, \ldots, N, \quad \text{for } \textit{constants } a_{ij}.$$

Then the requirement that the new basis also be L^2-orthonormal easily implies $AA^t = I$: A is orthogonal; and therefore $\sum_j |f_j(x)|^2 = \sum_i |e_i(x)|^2$, for all x.

In fact, the sum f has an intrinsic description, obtained by expressing orthogonal projection from sections of E to F as an integral operator:

$$(pr_F s)(x) = \sum_i \langle e_i, s\rangle_{L^2} e_i(x) = \sum_i \int_M \langle e_i(y), s(y)\rangle e_i(x) d\mu_M(y) = \int_M k(x,y)[s(y)] d\mu_M(y),$$

where the 'kernel' of pr_F (in the sense of integral operators, which is confusing terminology here) is:

$$k(x,y) = \sum_i e_i(y)^* \otimes e_i(x) \in \mathcal{L}(E_y, E_x).$$

The trace of k is defined as:

$$(trk)(x,y) = \sum_j \langle k(x,y)[e_j(y)], e_j(x)\rangle_{E_x} = \sum_{i,j} \langle e_j(y), e_i(y)\rangle_{E_y} \langle e_i(x), e_j(x)\rangle_{E_x}.$$

Then one easily computes:

$$\int_M (trk)(x,y) d\mu_M(y) = \sum_{i=1}^N |e_i(x)|^2 = f(x).$$

Main Lemma. If $F \subset C^\infty(E)$ is a finite-dimensional space of smooth sections, we have (with $l = \text{rank}(E)$):

$$\frac{\dim(F)}{l} \leq vol(M) \sup\{\frac{||s||_\infty^2}{||s||_{L^2}^2}; s \in F, s \not\equiv 0\}.$$

Proof. Let $x_0 \in M$ be a point of maximum for f; consider the evaluation map $ev_{x_0} : F \to E_{x_0}, s \mapsto s(x_0)$; let m be its rank, so $m \leq l$. Consider an L^2-orthonormal basis $\{f_1, \dots, f_m\}$ of $Ker(ev_{x_o})^\perp \subset F$, and complete it to an L^2-orthonormal basis (f_i) of F. Since f can also be computed in this basis, we have:

$$f(x_0) = \sum_{i=1}^m |f_i(x_0)|^2 \leq m \max_i \sup_{x \in M} |f_i|(x_0)^2$$

$$\leq l \sup\{||s||_\infty^2; s \in F, ||s||_{L^2} = 1\} = l \sup\{\frac{||s||_\infty^2}{||s||_{L^2}^2}; s \in F, s \not\equiv 0\}.$$

On the other hand, we have:

$$\dim(F) = \int_M f d\mu_M \leq vol(M) f(x_0).$$

This concludes the proof.

6. Proof of the main theorem.

We apply the main lemma to the space $\mathcal{H} = \text{Ker}(\Delta_H)$ of smooth sections of E, known to be finite-dimensional. By Lemma 3 (Kato inequalities) and the hypothesized lower bound on $\mathcal{R}_{min}$, if $s \in \mathcal{H}$, $|s|$ is, in the sense of distributions, a nonnegative solution of the inequality:

$$-\Delta|s| \leq \lambda^2|s|, \quad \lambda^2 = \frac{\Lambda^2}{diam(M)^2}$$

Thus Theorem 4 (Moser iteration) implies that if $s \in \mathcal{H}$ is non-zero,

$$\frac{||s||_\infty^2}{||s||_{L^2}^2} \leq \frac{B_n(x)}{vol(M)}, \quad x = \gamma vol(M)^{1/n}\lambda, \quad \lambda = \frac{\Lambda}{diam(M)}.$$

Here γ is the constant in the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$. By Theorem 1(iv) (Sobolev constant),

$$\gamma = vol(M)^{-1/n} R\sigma_n,$$

R given by Theorem 2 (Ricci control of isoperimetric profile): $R = \frac{diam(M)}{a(n,\alpha)}$. It follows that:

$$x = vol(M)^{-1/n}\frac{diam(M)}{a(n,\alpha)}\sigma_n vol(M)^{1/n}\frac{\Lambda}{diam(M)} = \frac{\sigma_n\Lambda}{a(n,\alpha)}$$

and, from the main lemma:

$$\frac{\dim(\mathcal{H})}{l} \leq B_n(\frac{\sigma_n\Lambda}{a(n,\alpha)}) := b(n,\alpha,\Lambda)$$

Since $B_n(x) \to 1$ as $x \to 0_+$, we may find $A = A(n,\alpha)$ so that if $\Lambda \leq A$, $\frac{\dim(\mathcal{H})}{l} < \frac{l+1}{l}$, and hence $\dim\mathcal{H} \leq l$, as we wished to show.