PART 1: BOCHNER METHOD WITH BOUNDARY AND TWISTED OPERATORS

**1. Hodge Theory on manifolds with boundary.** Let $(M^n, g)$ be a compact Riemannian manifold with nonempty boundary $\partial M$. At boundary points, the space of smooth $k$-forms on $M$ admits the direct sum decomposition:

$$\Omega^k(M)_{|\partial M} = \Omega^k(\partial M) \oplus (\nu^\# \wedge \Omega^{k-1}(\partial M)), \quad k \geq 1,$$

where $\nu$ is the outward unit normal and $\nu^\#$ the dual 1-form. Accordingly, any $\omega \in \Omega^k(M)$ admits a decomposition:

$$\omega = t(\omega) + n(\omega),$$

where, at points $p \in \partial M$, $t(\omega) \in \Omega^k(\partial M)$ (or more precisely, its image $i^*(t(\omega))$ under the pullback by the inclusion $i : \partial M \to M$ is), and $n(\omega) = \nu^\# \wedge \eta, i^*\eta \in \Omega^{k-1}(\partial M)$.

*Remark.* Note this decomposition of $\Omega^k(M)_{|\partial M}$ is orthogonal (with respect to the pointwise metric in $\Omega^k(M)$).

The usual 'Dirichlet' (or 'relative' boundary condition), $t(\omega) = 0$ and 'Neumann' (or 'absolute' boundary condition), $n(\omega) = 0$, define subspaces of $\Omega^k(M)$:

$$\Omega^k_D(M) = \{\omega \in \Omega^k(M); t(\omega) = 0 \text{ on } \partial M\}; \quad \Omega^k_N(M) = \{\omega \in \Omega^k(M); n(\omega) = 0 \text{ on } \partial M\}.$$

It is easy to check that the space $\Omega_D(M)$ is $d$-invariant, while $\Omega_N$ is $\delta$-invariant:

$$d : \Omega^k_D \to \Omega^{k+1}_D, \quad \delta : \Omega^k_N \to \Omega^{k-1}_N.$$

To see this, note $\nu^\#$ extends to a collar neighborhood of $\partial M$ as the exact 1-form $-d\rho$, where $\rho : M \to R_+$ is distance to $\partial M$, so that:

$$d(\nu^\# \wedge \eta) = -\nu^\# \wedge d\eta \text{ and } \delta(i_\nu \omega) = -i_\nu \omega. \quad (\omega \in \Omega^k_N \Leftrightarrow i_\nu \omega = 0 \text{ on } \partial M)$$

Thus we have differential complexes $(\Omega_D, d)$ (increasing degree) and $(\Omega_N, \delta)$ (decreasing degree), with (de Rham) cohomology spaces:

$$H^k_{rel}(M) = \frac{Ker(d_{|\Omega^k_D})}{Im(d_{|\Omega^{k-1}_D})}, \quad H^k_{abs}(M) = \frac{Ker(\delta_{|\Omega^k_N})}{Im(\delta_{|\Omega^{k+1}_N})}.$$

('Relative' resp. 'absolute' de Rham cohomology spaces.) The reason for this terminology is the *De Rham Theorem for manifolds with boundary*, which states:

$$H^k_{rel}(M) \approx H^k(M, \partial M), \quad H^k_{abs}(M) \approx H^k(M).$$

(Singular cohomology with $\mathbb{R}$ coefficients on the right.)

*Twisted de Rham complex.* At this point we introduce 'twisted' de Rham complexes, for an arbitrary smooth twisting function $f : M \to R$. The twisted differential $d_f$ and its $L^2$ adjoint $\delta_f$ are:

$$d_f = e^{-f} d e^f, \quad \delta_f = e^f \delta e^{-f}.$$

One checks easily that $d_f$ preserves $\Omega_D$ and $\delta_f$ preserves $\Omega_N$, so again we have two differential complexes, with cohomology spaces defined in the usual way. For instance, to see this for $\Omega_D$ note: $d_f = d + e_{\nabla f}$ (exterior product), and:

$$e_{\nabla f}(\nu^\# \wedge \eta) = df \wedge (\nu^\# \wedge \eta) = d^T f \wedge (\nu^\# \wedge \eta) = -\nu^\# \wedge (d^T f \wedge \eta),$$

where we define $d^T f := (df)_t$. And it's just as easy for the formal adjoint $\delta_f$ (using $\delta_f = \delta + i_{\nabla f}$).

We *claim* twisting doesn't change the absolute and relative de Rham cohomology spaces. To see this for the absolute cohomology, consider the isomorphism $\phi_a(\omega) = e^{-f}\omega$ from $\Omega^k_N$ to itself. This is in fact a chain isomorphism from the complex $(\Omega_N, \delta)$ to $(\Omega_N, \delta_f)$ since:

$$\phi_a(\delta_f \omega) = e^{-f}(e^f \delta e^{-f}\omega) = \delta(\phi_a \omega).$$

(The inverse is the chain map $\omega \mapsto e^f\omega$). Therefore $\phi_a$ induces isomorphisms in absolute de Rham cohomology: $H^k_{abs,f} \approx H^k_{abs}$, and henceforth we'll use just $H^k_{abs}$ for the cohomology space, also for the twisted complex. And likewise for $H^k_{rel} \approx H^k_{rel,f}$.

Naturally there are also 'twisted' Hodge Laplacians:

$$\Delta^f_H = d_f\delta_f + \delta_f d_f : \Omega^k \to \Omega^k.$$

To look for expressions relating $\Delta^f_H$ and $\Delta_H$, it is useful to introduce two Clifford actions on $\Omega(M)$:

$$c_X = e_X - i_X, \quad \tilde{c}_X = e_X + i_X, X \in TM.$$

It is easily checked that $c_X$ is skew-adjoint in $\Omega(M)$, while $\tilde{c}_X$ is symmetric.

They satisfy the following commutation relations:

$$c_X\tilde{c}_Y + \tilde{c}_Y c_X = 0, \quad \tilde{c}_X\tilde{c}_Y + \tilde{c}_Y\tilde{c}_X = 2\langle X, Y\rangle, \quad c_X c_Y + c_Y c_X = -2\langle X, Y\rangle,$$

for $X, Y \in TM$. Using the definitions, we find these are equivalent to:

$$e_X i_Y + i_Y e_X = \langle X, Y\rangle, \text{ in particular } e_X i_X + i_X e_X = |X|^2.$$

To compute an expression for $\Delta f$, we use (with summation convention, and $f_i = e_i(f)$) in an o.n. frame $(e_i)$, normal at a given $p \in M$:

$$d_f\delta_f\omega = -e_{e_i}(\nabla_{e_i} + f_i)i_{e_j}(\nabla_{e_j} - f_j)\omega, \quad \delta_f d_f\omega = -i_{e_i}(\nabla_{e_i} - f_i)e_{e_j}(\nabla_{e_j} + f_j)\omega.$$

Expanding, adding the results and using the commutation relations, we find:

$$\Delta^f_H\omega = \Delta_H\omega + |\nabla f|^2\omega + (\mathrm{Hess} f)(e_i, e_j)(e_{e_i}i_{e_j} - i_{e_i}e_{e_j})\omega.$$

Now use $i_{e_i}e_{e_j} = -e_{e_j}i_{e_i} + \delta_{ij}$ in the last term to conclude:

$$\Delta^f_H\omega = \Delta_H\omega + |\nabla f|^2\omega - (\Delta f)\omega + 2(\mathrm{Hess} f)(e_i, e_j)(e_{e_i}i_{e_j})\omega.$$

For the last term, we note that:

$$2\langle(\mathrm{Hess}f)(e_i,e_j)(e_{e_i}i_{e_j})\omega,\omega\rangle = 2(\mathrm{Hess}f)(e_i,e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle.$$

It is useful to know that the Dirichlet and Neumann subspaces of $\Omega^k$ admit simple descriptions in terms of the Clifford multiplications defined above. Namely, consider the operator:

$$\chi : \Omega^k_{|\partial M} \to \Omega^k_{|\partial M}, \quad \chi\omega := \tilde{c}_\nu c_\nu \omega = (i_\nu e_\nu - e_\nu i_\nu)\omega.$$

It is easy to show this is a self-adjoint, idempotent operator ($\chi^2 = Id$), hence diagonalizable with eigenvalues $\pm 1$.

**Lemma 1.** $\Omega_D = \{\omega; \chi\omega = -\omega \text{ on } \partial M\}$; $\quad \Omega_N = \{\omega; \chi\omega = \omega \text{ on } \partial M\}$.
*Proof.* An easy calculation shows that:

$$\chi(t(\omega)) = t(\omega), \quad \chi(n(\omega)) = -n(\omega),$$

and hence: $\chi(\omega) = \chi(t(\omega) + n(\omega)) = t(\omega) - n(\omega)$. Or we could note that $\Omega_D = \{\omega; e_\nu i_\nu \omega = \omega\}$ and $\Omega_N = \{\omega; i_\nu e_\nu \omega = \omega\}$.

*Question:* Do the operators $\Delta_H$ and $\Delta^f_H$ preserve $\Omega_D$ or $\Omega_N$?

Consider 1-forms first. Let $\alpha \in \Omega^1_D, t(\alpha) = 0, \alpha = g\nu^\# = gd\rho$, for some function $g$. Then:

$$d\alpha = dg \wedge d\rho, \quad \delta d\alpha = -(\Delta g)d\rho + (\Delta\rho)dg,$$

$$\delta\alpha = -\langle dg, d\rho\rangle, \quad d(\delta\alpha) = -\nu\langle dg, d\rho\rangle d\rho - [\mathrm{Hess}(g)(e_i, \nabla\rho) - A(e_i, \nabla^T g)]\theta_i,$$

where $A$ is the second fundamental form of $T(\partial M)$, $\langle\mathcal{W}(X), Y\rangle = A(X,Y) = \langle\nabla_X \nu, Y\rangle$, for $X, Y \in T(\partial M)$. Thus the tangential component of $\Delta_H\alpha$ is:

$$t(\Delta_H\alpha) = (\Delta\rho)d^T g + \mathrm{Hess}(g)(e_i,\nu)\theta_i + A(e_i, \nabla^T g)\theta_i,$$

not zero in general. Thus $\Omega_D$ is not preserved by $\Delta_H$.

In spite of this, there is a Hodge theory for $\Delta_H$ with boundary conditions $t(\omega) = 0$ or $n(\omega) = 0$. Namely, both are elliptic boundary conditions and the general Hodge decomposition theorem for elliptic complexes applies. Define spaces of harmonic $k$-forms:

$$\mathcal{H}^k_D = \{\omega \in \Omega^k_D; \Delta_H\omega = 0\}; \quad \mathcal{H}^k_N = \{\omega \in \Omega^k_N; \Delta_H\omega = 0\},$$

with similar definitions for the twisted Hodge Laplacian $\Delta^f_H$. Then we have unique representatives of relative (resp. absolute) de Rham cohomology in these spaces:

$$\mathcal{H}^K_D \approx \mathcal{H}^{k,f}_D \approx H^k_{rel}(M); \quad \mathcal{H}^k_D \approx \mathcal{H}^{k,f}_N \approx H^k_{abs}(M).$$

(Note $\Delta_H^f$ and $\Delta_H$ have the same principal symbol, as seen above.)

*Twisted Dirac operators.* In addition to the usual Dirac operator on $\Omega(M)$:

$$\mathcal{D} := d + \delta = \sum_i c_{e_i} \nabla_{e_i}, \quad \text{with } \mathcal{D}^2 = \Delta_H,$$

we have a twisted version:

$$\mathcal{D}_f = d_f + \delta_f = (d + e_{\nabla f}) + (\delta + i_{\nabla f}) = \mathcal{D} + \tilde{c}_{\nabla f}, \quad \text{with } \mathcal{D}_f^2 = \Delta_H^f.$$

The Dirac operator on $\Omega(M)$ satisfies the classical Weitzenböck formula:

$$\mathcal{D}^2 \omega = \Delta_H \omega = \nabla^* \nabla \omega + \mathcal{R}\omega,$$

with:

$$\nabla^* \nabla \omega = -\sum_i \nabla^2_{e_i, e_i} \omega, \quad \mathcal{R}\omega = \frac{1}{2} \sum_{i,j=1}^n c_{e_i} c_{e_j} R_{e_i, e_j} \omega.$$

We compute the version of the formula for the twisted Hodge Laplacian, in the pointwise quadratic form:

$$\langle \mathcal{D}_f^2 \omega, \omega \rangle = \langle \Delta_H^f \omega, \omega \rangle$$

$$= \langle \Delta_H \omega, \omega \rangle + (|\nabla f|^2 - \Delta f)|\omega|^2 + 2(\text{Hess}f)(e_i, e_j)\langle i_{e_i} \omega, i_{e_j} \omega \rangle$$

$$= \langle \nabla^* \nabla \omega, \omega \rangle + \langle \mathcal{R}\omega, \omega \rangle + [|\nabla f|^2 - (\Delta f)]|\omega|^2 + 2(\text{Hess}f)(e_i, e_j)\langle i_{e_i} \omega, i_{e_j} \omega \rangle$$

Next we compute the integrated twisted Weitzenböck formula with boundary term. Recall that for the untwisted Dirac operator we have, for $\omega \in \Omega^p$:

$$\int_M |\nabla \omega|^2 - |\mathcal{D}\omega|^2 + \langle \mathcal{R}\omega, \omega \rangle = \int_{\partial M} \langle \nabla_\nu \omega + c_\nu \mathcal{D}\omega, \omega \rangle = \int_{\partial M} \langle c_\nu \mathcal{D}^T \omega, \omega \rangle.$$

(For the last equality, consider a frame with $e_n = \nu$ and $e_i \in T(\partial M)$, $i = 1, \dots n-1$, and define $\mathcal{D}^T \omega = \sum_{i=1}^{n-1} c_{e_i} \nabla_{e_i} \omega$.)

The boundary term arises, on the one hand, from the fact that:

$$-\langle \nabla^* \nabla \omega, \omega \rangle + |\nabla \omega|^2 = \sum_j \langle \nabla^2_{e_j, e_j} \omega, \omega \rangle + |\nabla \omega|^2 = \sum_j e_j \langle \nabla_{e_j} \omega, \omega \rangle,$$

a divergence term. To compute the boundary term corresponding to formal self-adjointness of $\mathcal{D}_f$, consider (in a frame $(e_j)$ normal at a given point):

$$\langle \mathcal{D}_f \omega, \omega \rangle = \sum_j \langle c_{e_j} \nabla_{e_j} \omega, \omega \rangle + \langle \tilde{c}_{\nabla f} \omega, \omega \rangle$$

$$= \sum_j e_j \langle c_{e_j} \omega, \omega \rangle - \sum_j \langle c_{e_j} \omega, \nabla_{e_j} \omega \rangle + \langle \tilde{c}_{\nabla f} \omega, \omega \rangle$$

$$= \sum_j e_j \langle c_{e_j}\omega, \omega\rangle + \sum_j \langle \omega, c_{e_j}\nabla_{e_j}\omega\rangle + \langle \omega, \tilde{c}_{\nabla f}\omega\rangle$$

$$= div(X) + \langle \omega, \mathcal{D}_f \omega\rangle,$$

for a suitable vector field $X$. We conclude:

$$\int_M \langle \mathcal{D}_f \omega, \omega\rangle = \int_M \langle \omega, \mathcal{D}_f\omega\rangle + \int_{\partial M} \langle c_\nu \omega, \omega\rangle,$$

which implies:

$$\int_M \langle \mathcal{D}_f^2 \omega, \omega\rangle = \int_M |\mathcal{D}_f\omega|^2 + \int_{\partial M} \langle c_\nu \mathcal{D}_f\omega, \omega\rangle.$$

Thus the boundary term in the integrated Weitzenböck formula for $\mathcal{D}_f$ is:

$$\int_{\partial M} \langle \nabla_\nu \omega + c_\nu \mathcal{D}_f \omega, \omega\rangle.$$

The integrand can be simplified as before:

$$\langle \nabla_\nu \omega + c_\nu \mathcal{D}_f\omega, \omega\rangle = \langle c_\nu \mathcal{D}^T\omega, \omega\rangle + \langle c_\nu \tilde{c}_{\nabla f}\omega, \omega\rangle = \langle c_\nu \mathcal{D}_f^T \omega, \omega\rangle,$$

if we define $\mathcal{D}_f^T\omega := \mathcal{D}^T\omega + \tilde{c}_{\nabla f}\omega$.

The foregoing calculations prove the following lemma.

**Lemma 2.** *Integrated Weitzenböck formula for the twisted Dirac operator, with boundary term.*

$$\int_M |\nabla\omega|^2 - |\mathcal{D}_f\omega|^2 + \langle \mathcal{R}\omega, \omega\rangle + [|\nabla f|^2 - (\Delta f)]|\omega|^2 + 2\sum_{i,j} (\mathrm{Hess} f)(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle$$

$$= \int_{\partial M} \langle \nabla_\nu\omega + c_\nu \mathcal{D}_f\omega, \omega\rangle = \int_{\partial M} \langle c_\nu \mathcal{D}_f^T\omega, \omega\rangle.$$

To make use of this expression, two things are needed: (i) interpret the boundary term in terms of the geometry of the boundary; (ii) control the Hessian term.

Regarding the first point, we first consider untwisted operators, and $p$-forms satisfying Neumann boundary conditions. (I.e. $\chi\omega = \omega$.)

**Lemma 3.** Let $\omega \in \Omega^p_N, \chi\omega = \omega$. Then on $\partial M$:

$$(i) \langle c_\nu \mathcal{D}^T\omega, \omega\rangle = -\sum_{i,j} A(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle.$$

$$(ii) \langle c_\nu \tilde{c}_{\nabla f}\omega, \omega\rangle = -\nu(f)|\omega|^2.$$

*Proof. Step 1:* We show that, without assuming the Neumann condition:

$$\chi(c_\nu \mathcal{D}^T\omega) + c_\nu\mathcal{D}^T(\chi\omega) = -\sum_{i,j} A(e_i, e_j)c_{e_i}(\tilde{c}_{e_j} - c_{e_j}\chi)\omega.$$

Indeed, computing in a normal frame at $p \in M$ and using the summation convention, the left-hand side equals:

$$-\tilde{c}_\nu c_{e_i} \nabla_{e_i}\omega + c_\nu c_{e_i}(\tilde{c}(\nabla_{e_i}\nu)c_\nu\omega + \tilde{c}_\nu c(\nabla_{e_i}\nu)\omega + \tilde{c}_\nu c_\nu \nabla_{e_i}\omega)$$

$$= -(\tilde{c}_\nu c_{e_i} + c_{e_i}\tilde{c}_\nu)\nabla_{e_i}\omega + A(e_i, e_j)c_\nu c_{e_i}(\tilde{c}_\nu c_{e_j} + \tilde{c}_{e_j} c_\nu)\omega,$$

where the first term vanishes, and using the commutation relations we find:

$$\ldots = -A(e_i, e_j)c_{e_i}(\tilde{c}_{e_j} - c_{e_j}\chi)\omega,$$

as claimed.

*Step 2.* We show that, still without using the boundary condition:

$$\sum_{i,j} A(e_i, e_j)c_{e_i}(\tilde{c}_{e_j} - c_{e_j})\omega = 2\sum_{i,j} A(e_i, e_j)\theta_i \wedge i_{e_j}\omega.$$

Indeed, $(\tilde{c}_{e_j} - c_{e_j})\omega = 2i_{e_j}\omega$, while:

$$c_{e_i} i_{e_j}\omega = e_{e_i} i_{e_j}\omega - i_{e_i} i_{e_j}\omega,$$

and the second term will not contribute to the sum, since it is skew-symmetric in $i, j$. We conclude:

$$\sum_{i,j} A(e_i, e_j)c_{e_i}(\tilde{c}_{e_j} - c_{e_j})\omega = 2\sum_{i,j} A(e_i, e_j)e_{e_i} i_{e_j}\omega,$$

as claimed.

*Step 3.* Combining steps 1 and 2 and using the boundary condition $\chi\omega = \omega$ (and recalling $\chi$ is self-adjoint), we find:

$$\langle c_\nu \mathcal{D}^T\omega, \omega\rangle = (1/2)\langle \chi c_\nu \mathcal{D}^T\omega + c_\nu \mathcal{D}^T(\chi\omega), \omega\rangle$$

$$= -(1/2)\langle \sum_{i,j} A(e_i, e_j)c_{e_i}(\tilde{c}_{e_j} - c_{e_j})\omega, \omega\rangle$$

$$= -\langle \sum_{i,j} A(e_i, e_j)\theta_i \wedge i_{e_j}\omega, \omega\rangle = -\sum_{i,j} A(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle,$$

concluding the proof of (i).

To see (ii) for $p$-forms $\omega$ satisfying Neumann boundary conditions, recall $\omega = \chi\omega = \tilde{c}_\nu c_\nu\omega$. Then:

$$c_\nu \tilde{c}_{\nabla f}\omega = c_\nu \tilde{c}_{\nabla f}\tilde{c}_\nu c_\nu\omega = -\tilde{c}_{\nabla f}\tilde{c}_\nu\omega = -\tilde{c}_{\nabla f} e_\nu\omega.$$

Taking inner product with $\omega$, note $\langle e_{\nabla f} e_\nu\omega, \omega\rangle = 0$ (different degrees). Thus:

$$\langle c_\nu \tilde{c}_{\nabla f}\omega, \omega\rangle = -\langle i_{\nabla f} e_\nu\omega, \omega\rangle = -\langle e_\nu\omega, e_{\nabla f}\omega\rangle = -\nu(f)\langle e_\nu\omega, e_\nu\omega\rangle = -\nu(f)|\omega|^2.$$

The right-hand side of (i) can be estimated if the boundary is *p-convex*: the sum of the first $p$ smallest eigenvalues of the second fundamental form $A$ is nonnegative.

**Lemma 4.** Suppose $A$ has the property that the sum of any $p$ eigenvalues of $A$ is greater than or equal to a constant $(-\lambda) \in R$. Then, if $\omega \in \Omega^p_N$:

$$\sum_{i,j} A(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle \geq -\lambda|\omega|^2.$$

*Proof.* Let $\mathcal{I}_p$ be the set of increasing $p$-multitindices, $I = (i_1, \ldots i_p)$ with $i_1 < \ldots < i_p$. Then if $(e_i)$ is a local orthonormal frame on $T(\partial M)$ with dual coframe $(\theta_i)$, we have $\omega = \sum_{I \in \mathcal{I}_p} \omega_I \theta_I$. Choose $(e_i)$ to diagonalize $A$: $A(e_i, e_j) = \lambda_i \delta_{ij}$. Then:

$$\sum_{I,J \in \mathcal{I}_p} A(e_k, e_l)\omega_I \bar{\omega_J}\langle i_{e_k}\theta_I, i_{e_l}\theta_J\rangle$$

$$= \sum_{I,J} \lambda_k \omega_I \bar{\omega}_J \langle i_{e_k}\theta_I, i_{e_k}\theta_J\rangle,$$

where we note $\langle i_{e_k}\theta_I, i_{e_k}\theta_J\rangle$ is nonzero only if $I = J$ and $k \in I$. Thus:

$$\sum_{k,l}\sum_{I,Jl} A(e_k, e_l)\omega_I \bar{\omega_J}\langle i_{e_k}\theta_I, i_{e_l}\theta_J\rangle = \sum_k \sum_{I \in \mathcal{I}_p; k \in I} \lambda_k |\omega_I|^2$$

$$= \sum_I (\lambda_{i_1} + \ldots + \lambda_{i_p})|\omega_I|^2 \geq -\lambda|\omega|^2,$$

if $A$ has the property given in the statement.

Combining the previous two lemmas, we have a simple inequality for the boundary term in the integrated Weitzenböck formula for $\omega \in \Omega^p_N$, when the boundary is $p$-convex.

**Corollary 1.** Assume the second fundamental form of $\partial M$ has the property that the sum of any $p$ eigenvalues is bounded below by a fixed real number $\lambda$. Then if $\omega \in \Omega^p_N$:

$$\langle c_\nu \mathcal{D}^T_f \omega, \omega\rangle \leq [\lambda - \nu(f)]|\omega|^2.$$

We now turn to the Hessian term in the integrated Weitzenböck formula. Computing in an orthonormal frame $(e_i)$, normal at some $p \in M$:

$$\sum_{i,j} \mathrm{Hess}(f)(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle$$

$$= \sum_{i,j} e_i[(\nabla_{e_j} f)\langle i_{e_i}\omega, i_{e_j}\omega\rangle] - \sum_{i,j}(\nabla_{e_j} f)\langle i_{e_i}\nabla_{e_i}\omega, i_{e_j}\omega\rangle - \sum_{i,l}(\nabla_{e_j} f)\langle i_{e_i}\omega, i_{e_j}\nabla_{e_i}\omega\rangle$$

$$= \mathrm{div} Z + \langle \delta\omega, i_{\nabla f}\omega\rangle - \sum_{i,j} (\nabla_{e_j} f)\langle \omega, \theta_i \wedge i_{e_j})\nabla_{e_i}\omega\rangle,$$

where $Z$ is the vector field dual to the one-form $X \mapsto \langle i_X\omega, i_{\nabla f}\omega\rangle$. Using now $\langle \omega, \theta_i \wedge i_{e_j})\nabla_{e_i}\omega = (\delta_{ij} - i_{e_j} e_{e_i})\nabla_{e_i}\omega$, we conclude:

$$... = \mathrm{div} Z + \langle \delta\omega, i_{\nabla f}\omega\rangle - \langle \omega, \nabla_{\nabla f}\omega\rangle + \langle df \wedge \omega, d\omega\rangle.$$

This is already interesting: in complete generality, the Hessian term reduces, up to a divergence, to geometric first-order terms.

Now suppose $\Delta^f_H\omega = 0$. Then $\delta_f\omega = 0$ and $d_f\omega - 0$, that is: $\delta\omega = -i_{\nabla f}\omega, d\omega = -df \wedge \omega$. Substituting in the above, we find:

$$\langle \delta\omega, i_{\nabla f}\omega\rangle + \langle df \wedge \omega, d\omega\rangle = -|i_{\nabla f}\omega|^2 - |df \wedge \omega|^2 = -|\nabla f|^2|\omega|^2.$$

We conclude:

**Lemma 5.** Suppose $\omega \in \mathcal{H}^p_{N,f}$ or $\omega \in \mathcal{H}^p_{D,f}$. Then:

$$\sum_{i,j} \mathrm{Hess}(f)(e_i, e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle = -\int_M |\nabla f|^2|\omega|^2 - \int_M \langle \omega, \nabla_{\nabla f}\omega\rangle + \int_{\partial M} \langle i_\nu\omega, i_{\nabla f}\omega\rangle.$$

*Remark 1:* Note that the last term vanishes if $\omega \in \Omega^p_N$.

*Remark 2:* The sum of the first two terms is bounded below by:

$$-\int_M \frac{3}{2}|\nabla f|^2|\omega|^2 - \int_M \frac{1}{2}|\nabla\omega|^2.$$

PART 2: POSITIVE ISOTROPIC CURVATURE.

**Definitions.** Let $(M, g)$ be a Riemannian manifold, where we also denote the Riemannian metric by $\langle\cdot,\cdot\rangle$. There are two natural ways to extend the metric to the complexified tangent bundle, $TM^c := TM \otimes \mathbb{C}$. We can extend it as a symmetric, $\mathbb{C}$-bilinear form: if $z = x + iy, w = u + iv$ are in $T_pM^c$ (with $x, y, u, v \in T_pM$), set:

$$(z, w) = (x + iy, u + iv) := \langle x, u\rangle - \langle y, v\rangle + i[\langle y, u\rangle + \langle x, v\rangle].$$

Or we can extend it as a hermitian inner product (conjugate-linear in the second entry), by setting:

$$\langle\langle z, w\rangle\rangle := (z, \bar{w}) = \langle x, u\rangle + \langle y, v\rangle + i[\langle y, u\rangle - \langle x, v\rangle].$$

Similarly, the induced inner product on each exterior bundle $\Lambda^k T^*M$ extends in two ways to its complexification $\Lambda^k_c(M) = \Lambda^k T^*M \otimes \mathbb{C}$.

Recall that the *curvature operator* is the symmetric linear operator $\mathbf{R}$ defined on $\Lambda^2 TM$ in therms of the $(3,1)$ curvature tensor $R$ by:

$$\langle \mathbf{R}(x \wedge y), u \wedge v \rangle := \langle R(x,y)v, u \rangle, \quad x, y, u, v \in T_pM.$$

(Note the order, which corresponds to the convention that sectional curvatures are diagonal components of $\mathbf{R}$.) This naturally extends to a $\mathbb{C}$-linear, self-adjoint operator (for the hermitian metric) $\mathbf{R}$ on $\Lambda^2_c(M)$. We use it to define the *hermitian sectional curvature* $K^c$ of a complex two-dimensional subspace $\sigma \subset T_p^c M$: if $\{z, w\}$ is a basis for $\sigma$,

$$K^c(\sigma) := \langle\langle \mathbf{R}(z \wedge w), z \wedge w \rangle\rangle / ||z \wedge w||^2.$$

(where in the denominator we also use the hermitian inner product.)

To express this in Riemannian terms, we expand it (with $z = x + iy, w = u + iv$) to obtain:

$$\langle\langle \mathbf{R}(z \wedge w, z \wedge w \rangle\rangle = \langle \mathbf{R}(x \wedge u - y \wedge v), x \wedge u - y \wedge v \rangle + \langle \mathbf{R}(x \wedge v + y \wedge u), x \wedge v + y \wedge u \rangle,$$

a real number. Expanding further, using the definition of $\mathbf{R}$, we find in terms of the $(4,0)$ curvature:

$$... = \langle R(x,u)u, x \rangle + \langle R(y,v)v, y \rangle + \langle R(x,v)v, x \rangle + \langle R(y,u)u, y \rangle - 2\langle R(x,u)v, y \rangle + 2\langle R(y,u)v, x \rangle,$$

where (if $x, y, u, v$ happen to be orthornormal) the first four terms are (real) sectional curvatures, while the last two equal:

$$-2\langle R(x,u)v, y \rangle + 2\langle R(x,v)u, y \rangle = 2\langle R(u,x)v, y \rangle + 2\langle R(x,v)u, y \rangle = -\langle R(v,u)x, y \rangle = \langle R(x,y)u, v \rangle,$$

by the algebraic Bianchi identity. We conclude that, if $\{x, y, u, v\}$ is real-orthonormal, the hermitian sectional curvature of $\sigma$ is the real number:

$$K^c(\sigma) = K_{x,u} + K_{x,v} + K_{y,u} + K_{y,v} - 2R(x,y,u,v).$$

A condition guaranteeing orthonormality of the real and imaginary parts of a complex basis is the following.

*Definition.* A vector $z \in T_pM^c$ is *isotropic* if $(z, z) = 0$ (using the $\mathbb{C}$-bilinear form.) A subspace $\sigma \subset T_pM^c$ is *totally isotropic* if every vector in $\sigma$ is. Note that in terms of the real and imaginary parts this means:

$$|x|^2 = |y|^2, \quad \langle x, y \rangle = 0, \quad z = x + iy, x, y \in T_pM.$$

*Definition.* $(M, g)$ has *positive sectional curvature on isotropic two-planes* (in short: 'positive isotropic curvature', PIC) if $K^c(\sigma) > 0$ whenever $\sigma \subset T_p^c M$ is an isotropic complex-two-dimensional subspace.

To understand what this means in Riemannian terms, let $\sigma \subset T_pM^c$ be a complex two-dimensional subspace. We may choose a *standard basis* $\{z, w\}$ for $\sigma$, one satisfying, for the hermitian inner product:

$$||z||^2 = ||w||^2 = 2; \quad \langle\langle z, w \rangle\rangle = 0.$$

*Exercise.* Show that a standard basis $\{z, w\}$ of a (complex) two-dimensional subspace $\sigma \subset T_pM^c$ has the property that the real and imaginary parts: $z = e_1 + ie_2, w = e_3 + ie_4$ of $z, w$ define a Riemannian-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for a (real) four-dimensional subspace of $T_pM$ if, and only if, $\sigma$ is totally isotropic.

*Hint:* In addition to $(z, z) = (w, w) = 0$, use also $(z + w, z + w) = 0$, which implies $(z, w) = 0$.

Thus, for an isotropic complex two-plane $\sigma \subset T_pM^c$:

$$K^c(\sigma) = K_{e_1,e_3} + K_{e_1,e_4} + K_{e_2,e_3} + K_{e_2,e_4} - 2R(e_1, e_2, e_3, e_4),$$

in terms of a 'standard basis' $\{z, w\}$ for $\sigma$ and its real and imaginary parts $z = e_1 + ie_2, w = e_3 + ie_4$. Equivalently, with the notation $R_{ijkl} := \langle R(e_i, e_j)e_k, e_l\rangle$:

$$K^c(\sigma) = R_{1331} + R_{1441} + R_{2332} + R_{2442} - 2R_{1234}.$$

The following proposition relates the hermitian sectional curvature $K^c(\sigma)$ of isotropic 2-planes $\sigma \subset T_p^cM$ to the Weitzenböck curvature operator $\mathcal{R}$ on 2-forms.

**Proposition 1.** Assume the dimension of $M$ is even, $n = 2m \geq 4$. Let $\omega \in \Lambda^2(T^*M)^c$. Then:

$$\langle \mathcal{R}\omega, \omega\rangle \geq (m-1)|\omega|^2 \min\{K^c(\sigma); \sigma \subset T_pM^c \text{ isotropic complex } 2-\text{plane}\},$$

*Proof.* The proof is based on the following facts:

(a) The Weitzenböck curvature operator $\mathcal{R}$ on exterior forms $\omega \in \Lambda^k(T^*M)$ admits the alternative expression:

$$\mathcal{R}\omega = \sum_{i,j} \theta_i \wedge i_{e_j} R_{e_i e_j}\omega.$$

(b) The curvature tensor acts on 2-forms $\omega$ as a derivation, as follows:

$$(R_{X,Y}\omega)(Z, W) = -\omega(R_{X,Y}Z, W) - \omega(Z, R_{X,Y}W).$$

(c) There is a canonical isomorphism $\Lambda^2(T^*M)^c \approx \mathfrak{so}(2m, \mathbb{C})$, defined by $L_{\xi\wedge\eta}(X) = \xi(X)\eta^\# - \eta(X)\xi^\#$. Elements of the Lie algebra $\mathfrak{so}(2m, \mathbb{C})$ admit a standard block-diagonal form. Geometrically this means that given $\omega \in \Lambda^2(T^*M)^c$, we may find a real orthonormal frame $(e_i)_{i=1}^{2m}$, with coframe $(\theta_i)_{i=1}^{2m}$, which puts $\omega$ in 'standard form', that is, at any $p \in M$ there exist coefficients $\omega_i(p) \in \mathbb{C}$ so that:

$$\omega(p) = \sum_{i=1}^{m} \omega_i(p)\theta_{2i-1} \wedge \theta_{2i}.$$

(This is where the fact $n$ is even is used crucially.)

To understand this computation, consider first the case $n = 4, m = 2$. Let the 2-form $\omega$ have the representation (at a given $p \in M$):

$$\omega = \omega_1 \theta_1 \wedge \theta_2 + \omega_2 \theta_3 \wedge \theta_4, \quad \omega_1, \omega_2 \in \mathbb{C}.$$

Then using (a) and (b) one finds:

$$\langle \mathcal{R}(\theta_1 \wedge \theta_2), \theta_1 \wedge \theta_2 \rangle = R_{1331} + R_{1441} + R_{2332} + R_{2442},$$

$$\langle \mathcal{R}(\theta_3 \wedge \theta_4), \theta_3 \wedge \theta_4 \rangle = R_{3113} + R_{3223} + R_{4114} + R_{4224},$$

while

$$\langle \mathcal{R}(\theta_1 \wedge \theta_2), \theta_3 \wedge \theta_4 \rangle = 2R_{1234}, \quad \langle \mathcal{R}(\theta_3 \wedge \theta_4), \theta_1 \wedge \theta_2) \rangle = 2R_{3412}.$$

Using the algebraic symmetries of the Riemann (4,0) curvature $R$, we easily compute from this:

$$\langle \mathcal{R}\omega, \omega \rangle = (|\omega_1|^2 + |\omega_2|^2)(R_{1331} + R_{1441} + R_{2332} + R_{2442}) - [\omega_1 \bar{\omega}_2 + \omega_2 \bar{\omega}_1] 2R_{1234}.$$

$$= (|\omega_1|^2 + |\omega_2|^2)(R_{1331} + R_{1441} + R_{2332} + R_{2442} - 2R_{1234}) + |\omega_1 - \omega_2|^2 2R_{1234},$$

where the second term is nonnegative if $R_{1234} \geq 0$ (which may always be assumed by relabeling). We conclude, in this case ($n = 4$):

$$\langle \mathcal{R}\omega, \omega \rangle \geq K^c(\sigma)|\omega|^2,$$

where $\sigma$ is the totally isotropic complex 2-plane spanned by $\{e_1 + ie_2, e_3 + ie_4\}$.

In the general case ($n = 2m \geq 4$), a similar calculation (see [1]) yields the result:

$$\langle \mathcal{R}\omega, \omega \rangle \geq \sum_{i=1}^{m} (|\omega_i|^2 \sum_{j=1, j \neq i}^{m} K^c(\sigma_{ij})),$$

where $\sigma_{ij}$ is the isotropic 2-plane spanned (over $\mathbb{C}$) by $\{e_{2i-1} + \sqrt{-1} e_{2i}, e_{2j-1} + \sqrt{-1} e_{2j}\}$ (a 'standard basis' of $\sigma_{ij}$, in the sense defined above). We conclude that, pointwise on $M$:

$$\langle \mathcal{R}\omega, \omega \rangle \geq (m-1)|\omega|^2 \min\{K^c(\sigma); \sigma \subset T_p M^c \text{ isotropic complex } 2-\text{plane}\},$$

with equality achieved in some cases (i.e. this lower bound is 'sharp'.)

*Question:* What estimate do we get if $\dim(M)$ is odd?

Combining all the foregoing results, we obtain the following:

**Omnibus Lemma 6.** Suppose $(M^n, g)$ is a compact manifold with boundary, satisfying:

(i) $n = 2m$ is even, and the hermitian sectional curvature $K^c(\Pi) \geq \sigma$ for each isotropic complex 2-plane $\Pi \subset T_p M^c$;

(ii) The second fundamental form of $\partial M$ satisfies $A(X,X)+A(Y,Y)\geq -\delta$, for each $\{X,Y\}$ orthonormal vector fields tangent to $\partial M$.

Given $f : M \to R$ smooth, let $\omega \in \mathcal{H}^2_{N,f}$ be an $f$-harmonic 2-form with Neumann boundary conditions. (In particular, $\mathcal{D}_f\omega = 0$.)

Then we have:

$$\begin{aligned}
0 = \int_M |\nabla\omega|^2 + \langle \mathcal{R}\omega,\omega\rangle + [|\nabla f|^2 - (\Delta f)]|\omega|^2 \\
+2\sum_{i,j}(\mathrm{Hess} f)(e_i,e_j)\langle i_{e_i}\omega, i_{e_j}\omega\rangle - \int_{\partial M}\langle c_\nu \mathcal{D}_f^T\omega,\omega\rangle \\
\geq \int_M |\nabla\omega|^2 + (m-1)\sigma\int_M|\omega|^2 + \int_M [|\nabla f|^2-(\Delta f)]|\omega|^2 \\
-\int_M 3|\nabla f|^2|\omega|^2 - \int_M|\nabla\omega|^2 + \int_{\partial M}[\nu(f)-\delta]|\omega|^2. \\
= \int_M [(m-1)\sigma - (\Delta f) - 2|\nabla f|^2]|\omega|^2 + \int_{\partial M}(\nu(f)-\delta)|\omega|^2
\end{aligned}$$

(cp. [1], (4.2))