1. Clifford Algebras and the Clifford Bundle

Def. Let (V, g) be an n-dimensional vector space over $\mathbb{R}$ with a positive definite inner product g. The Clifford algebra $\mathrm{Cl}(V, g)$ is the associative algebra with unit 1 generated by $e_1, \ldots, e_n$ (an orthonormal basis of V) with the relations:

$$e_i^2 = -1, \quad e_i e_j + e_j e_i = 0, \quad \forall i \neq j.$$

Existence. In the tensor algebra $\mathcal{T}(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$, let $\mathcal{I}_g$ be the 2-sided ideal generated by elements of the form $v \otimes w + w \otimes v + 2g(v, w)$ for $v, w \in V$. Then we define:

$$\mathrm{Cl}(V, g) = \mathcal{T}(V)/\mathcal{I}_g.$$

Remark. This is isomorphic to the Clifford algebra of $\mathbb{R}^n$, denoted Cl_n, so we use both notations interchangeably.

1.1. **Universal Property.** Let $\mathcal{A}$ be an associative algebra with unit, and let $f : V \to \mathcal{A}$ be a linear map. Then f extends uniquely to an algebra homomorphism $\tilde{f} : \mathrm{Cl}(V) \to \mathcal{A}$, provided f satisfies:

$$f(v) \cdot f(v) = -g(v, v) \cdot 1, \quad \forall v \in V.$$

Remark. Note the linear map $\pi : V \to \mathrm{Cl}(V)$, composition of $V \hookrightarrow \mathcal{T}(V) \to \mathrm{Cl}(V)$ is injective since $\pi(v) \cdot \pi(v) = -g(v, v) \cdot 1$. So we identify V with its image in $\mathrm{Cl}(V)$.

1.2. **Canonical Vector Space Isomorphism.** There exists an isomorphism of vector spaces:

$$\Lambda^* V \cong \mathrm{Cl}(V),$$

where $\Lambda^* V$ is the exterior algebra. Note that $\mathrm{Cl}(V)$ is generated by products $e_{i_1} \cdots e_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq n$ (as a vector space), together with 1. The set of products$(e_{i_1} \cdots e_{i_p})$ forms a vector space basis for $\mathrm{Cl}(V)$, and mapping $(e_{i_1} \cdots e_{i_p}) \mapsto e_{i_1} \wedge \ldots \wedge e_{i_p}$ establishes a bijection between basis elements of $\Lambda^* V$ and of $\mathrm{Cl}(V)$. This defines the isomorphism. Both spaces have dimension 2^n as real vector spaces.

Remark. This is **not** an algebra homomorphism!

Lemma. Under this isomorphism:

$$v \cdot \varphi \mapsto v \wedge \varphi - i(v)\varphi, \quad \forall v \in V, \, \varphi \in \mathrm{Cl}(V) \approx \Lambda^* V.$$

Proof. Let $\{e_i\}$ be an orthonormal basis for V with $v = e_1$. Then

$$\begin{aligned} v \wedge \varphi &= e_i \wedge e_1 \wedge \cdots \wedge e_{i_p} \\ &= -e_{i_2} \cdot \ldots \cdot e_{i_p} \mapsto e_{i_2} \wedge \ldots \wedge_{i_p} = (ext(e_1) - int(e_1))\varphi, \quad \text{if } i_1 = 1 \\ &= e_1 \cdot e_{i_1} \ldots \cdot e_{i_p} \mapsto e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_p} = (ext(e_1) - int(e_1))\varphi, \quad \text{if } i_1 > 1. \end{aligned}$$

1.3. **Even/Odd Splitting.** From the universal property, there exists a homomorphism $\alpha : \mathrm{Cl}_n \to \mathrm{Cl}_n$ extending $v \mapsto -v$, which is an involution. This defines a splitting:

$$\mathrm{Cl}_n = \mathrm{Cl}_n^+ \oplus \mathrm{Cl}_n^-, \quad (+1 \text{ resp. } -1 \text{ eigenspaces of } \alpha).$$

The isomorphism $\Lambda^* V \cong \mathrm{Cl}(V)$ maps $\Lambda^{\mathrm{even}}(V)$ to Cl_n^+ isomorphically.
Note $\mathrm{Cl}_n^+ \cdot \mathrm{Cl}_n^+ \subset \mathrm{Cl}_n^+$: this is a subalgebra of Cl_n

Lemma. $\mathrm{Cl}_n \approx \mathrm{Cl}_{n+1}^+$.

Proof. Define $\psi : \mathrm{Cl}_n \to \mathrm{Cl}_{n+1}^+$ by:

$$\psi(a) = a^+ + a^- \cdot e_{n+1}, \quad \text{where } a = a^+ + a^- \in \mathrm{Cl}_n^+ \oplus \mathrm{Cl}_n^-.$$

Clearly $\psi(a) \in \mathrm{Cl}_{n+1}^{+}$, and one checks easily that this is an algebra isomorphism. For instance:

$$\begin{aligned}\psi(a\cdot b) &= (a\cdot b)^{+} + (a\cdot b)^{-}\cdot e_{n+1}, \text{ while}\\ \psi(a)\cdot\psi(b) &= (a^{+} + a^{-}\cdot e_{n+1})\cdot(b^{+} + b^{-}\cdot e_{n+1})\\ &= a^{+}\cdot b^{+} + a^{-}\cdot b^{-} + (a^{+}\cdot b^{-} - a^{-}\cdot b^{+})\cdot e_{n+1}\end{aligned}$$

and these coincide (as the reader may check).

1.4. **Action of the Orthogonal Group.** The orthogonal group O_n acts on Cl_n, in various ways. This is induced by the action on the tensor algebra. For example, the left action on $v\otimes w$ is given by $A(v\otimes w) = Av\otimes Aw$; it extends the standard action of O_n on $\mathbb{R}^n$ by left multiplication of 'column vectors'. To see this induces an action on Cl_n, It suffices to check that this action preserves the ideal $\mathcal{I}_g$:

$$A(v\otimes w + w\otimes v + 2g(v,w)1) = Av\otimes Aw + Aw\otimes Av + 2g(Av,Aw)1 \in \mathcal{I}_g.$$

(We define $A1 = 1$.) We can also use the action of O_n by right matrix multiplication of 'row vectors', $v\mapsto vA^t$ (we use A^t to turn it into a left action). This extends to the tensor algebra of $\mathbb{R}^n$, and induces a different action of O_n on Cl_n, $\psi\mapsto\psi A^t$.

Combining the two, we define the 'adjoint action' of O_n on Cl_n:

$$Ad(A)\psi = A\psi A^t, \quad A\in O_n, \psi\in \mathrm{Cl}_n.$$

(Note $A^t = A^{-1}$, for $A\in O_n$.)

1.5. **Classification and Complexification.** It is easy to see the isomorphisms (over $\mathbb{R}$):

$$\mathrm{Cl}_1 \approx \mathbb{C}, \quad \mathrm{Cl}_2 \approx \mathbb{H} \text{ (quaternions).}$$

We also have the periodicity: $\mathrm{Cl}_{n+8} \approx \mathrm{Cl}_n \otimes \mathrm{Cl}_8$.

The classification of *complexified* Clifford algebras is more regular. Define:

$$\mathbb{C}l_n = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C} = \{\psi + i\varphi; \psi,\varphi \in Cl_n\}.$$

Then we have (cp. Table 1, [L-M p. 28]):

$$\mathbb{C}\mathrm{l}_1 \cong \mathbb{C}\oplus\mathbb{C}, \quad \mathbb{C}\mathrm{l}_2 \cong \mathbb{C}(2) \quad \text{(2x2 complex matrices),}$$

and in general:

$$\mathbb{C}\mathrm{l}_{2k} \cong \mathbb{C}(2^k), \quad \mathbb{C}\mathrm{l}_{2k+1} \cong \mathbb{C}(2^k)\oplus\mathbb{C}(2^k).$$

(Check that the complex dimensions coincide.)

Also in the real case: Cl_n is isomorphic (over $\mathbb{R}$) to a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (quaternions), or to a direct sum of two such matrix algebras.

This leads to the classification of irreducible representations of $\mathbb{C}l_n$, since $\mathbb{C}(m)$ has a unique irreducible representation (the usual one over $\mathbb{C}^m$), while $\mathbb{C}(m)\oplus\mathbb{C}(m)$ has two inequivalent ones.

1.6. **Representations: Clifford Modules.** A complex representation of Cl_n is an algebra homomorphism (over $\mathbb{R}$):

$$\rho : \mathrm{Cl}_n \to \mathcal{L}(W),$$

where W is a finite-dimensional complex vector space (called a "Clifford module" or Cl_n-module).
Thus Cl_n acts on W via linear transformations, denoted as "Clifford multiplication" by $\varphi \in \mathrm{Cl}_n$:

$$\varphi \cdot w = \rho(\varphi) w \quad \forall w \in W.$$

W is *reducible* if $W = W_1 \oplus W_2$, with $\rho(\varphi)W_1 \subseteq W_1\ \forall\varphi$ and $\rho(\varphi)W_2 \subseteq W_2\ \forall\varphi$. Any such ρ induces a $\mathbb{C}$-homomorphism $\rho_c : \mathbb{C}\mathrm{l}_n \to \mathcal{L}(W)$ (complexification), and we have: ρ irreducible implies ρ_c irreducible, but not conversely.
Theorem. The number of irreducible (inequivalent) complex representations of Cl_n is 1, if n is even (of dimension 2^k if $n = 2k$) and 2, if n is odd (both of complex dimension 2^k, if $n = 2k+1$).
Example. It is possible to describe the complex irreducible representation of Cl_{2n} explicitly. The usual Hermitian inner product on $\mathbb{C}^n$ induces a contraction map $int(v) = i(v) : \Lambda^p(\mathbb{C}^n) \to \Lambda^{p-1}(\mathbb{C}^n)$. Now define:

$$f_v : \Lambda^*(\mathbb{C}^n) \to \Lambda^*(\mathbb{C}^n), \quad f_v(\omega) = v \wedge \omega - i(v)\omega.$$

Note that $\mathrm{ext}(v)\mathrm{ext}(v) = 0$, $\mathrm{int}(v)\mathrm{int}(v) = 0$, and $(\mathrm{ext}(v)\mathrm{int}(v) + \mathrm{int}(v)\mathrm{ext}(v))\varphi = -\|v\|^2\varphi$, so $f_v \circ f_v = -\|v\|^2\mathbb{I}$ on $\Lambda^*(\mathbb{C}^n)$.

Now write $v \in \mathbb{C}^n$ as $v = v_1 + iv_2$, where $(v_1, v_2) \in \mathbb{R}^{2n}$. It is easy to see that, for the euclidean norm in $\mathbb{R}^{2n}$ and the hermitian norm in $\mathbb{C}^n$, we have $||(v_1, v_2)|| = ||v||$. Thus the above conclusion may be written in the form $f_{(v_1,v_2)} \circ f_{(v_1,v_2)} = -||(v_1, v_2)||^2\mathbb{I}$, and therefore the map $(v_1, v_2) \mapsto f_{(v_1,v_2)}$ extends to an algebra homomorphism $\mathrm{Cl}_{2n} \to \mathcal{L}(\Lambda^*(\mathbb{C}^n))$ ($\mathbb{R}$-linear in $\varphi \in \mathrm{Cl}_{2n}$, $\mathbb{C}$-linear in $\omega \in \Lambda^*(\mathbb{C}^n)$). For dimensional reasons, this must be the unique (up to isomorphism) complex irreducible representation of Cl_{2n}.

1.7. **Invariant Inner Products.** The following important property holds:

Proposition. Let $\mathrm{Cl}_n \to \mathcal{L}(W)$ be a real representation. Then there exists a positive definite inner product on W such that Clifford multiplication by unit vectors $e \in \mathbb{R}^n$ is orthogonal. (If W is a complex Cl_n-module, the same conclusion holds for Hermitian inner products.)
Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and consider the subgroup $F_n \subseteq \mathrm{Cl}_n^\times$ (the group of invertible elements of Cl_n) generated by $\{e_1, \ldots, e_n\}$ and (-1), subject to the relations:

$$(-1) \text{ is central ;} \quad (-1)^2 = Id; \quad e_i^2 = -1 \quad e_ie_j = (-1)e_je_i \text{ if } i \neq j.$$

Note this is a finite group! So given any inner product on W, we can construct a second one invariant under F_n by averaging over this finite group.

Then if $e \in \mathbb{R}^n$ is a unit vector, $e = \sum x_ie_i$ with $x_i \in \mathbb{R}$, $\sum x_i^2 = 1$, we have:

$$\langle e\cdot w, e\cdot w\rangle = \sum_j x_j^2\langle e_j\cdot w, e_j\cdot w\rangle + \sum_{i\neq j} x_ix_j\langle e_i\cdot w, e_j\cdot w\rangle = (\sum_j x_j^2)\langle w, w\rangle = \langle w, w\rangle,$$

noting that $\langle e_j\cdot w, e_j\cdot w\rangle = \langle w, w\rangle$ ($e_j \in F_n$) and the second sum vanishes (symmetric times antisymmetric in $i \neq j$):

$$\langle e_i\cdot w, e_j\cdot w\rangle = -\langle w, e_i\cdot e_j\cdot w\rangle = \langle w, e_j\cdot e_i\cdot w\rangle = -\langle e_j\cdot w, e_i\cdot w\rangle.$$

(Here we already used the corollary below, applied to F_n.)

Remark: Since the e_i generate Cl_n multiplicatively, a representation of Cl_n is the same as a representation of F_n such that $-1 \mapsto -\mathbb{I}$.

Corollary. If $v \in \mathbb{R}^n$, Clifford multiplication by v is skew-symmetric in W (for any invariant inner product):

$$\langle v \cdot w, w' \rangle = -\langle w, v \cdot w' \rangle.$$

Proof. We may assume $v \neq 0$, so:

$$\langle v \cdot w, w' \rangle = \langle \frac{v}{||v||} \cdot v \cdot w, \frac{v}{||v||} \cdot w' \rangle = \frac{1}{||v||^2} \langle v \cdot v \cdot w, v \cdot w' \rangle = -\langle w, v \cdot w' \rangle,$$

since $v \cdot v = -||v||^2 1$.

1.8. **An application to Differential Topology.** If $\mathbb{R}^{N+1}$ is a Cl_n-module, there exist n vector fields on S^N, linearly independent at each point (i.e., $\mathrm{rank}(S^n) \geq n$).

Proof. Choose an invariant inner product on $\mathbb{R}^{N+1}$, making Clifford multiplication by unit vectors in $\mathbb{R}^n$ orthogonal maps. Then use Clifford multiplication to define, for each $v \in \mathbb{R}^n$ and each $x \in S^N$, the unit sphere in $\mathbb{R}^{N+1}$:

$$V(x) = v \cdot x,$$

Note $\langle v \cdot x, x \rangle = 0$ (since Clifford multiplication by v is skew-symmetric in $\mathbb{R}^{N+1}$), so $V(x) \in T_x S^N$. Thus we have defined a linear map from $\mathbb{R}^n$ to the space $\chi(S^N)$ of vector fields on the N-sphere. We claim that for each $x \in S^N$, the linear map $v \mapsto V(x)$ from $\mathbb{R}^n$ to $T_x S^N$ has trivial kernel, and therefore has rank n. Indeed:

$$V(x) = v \cdot x = 0 \Rightarrow v \cdot v \cdot x = 0 \Rightarrow -||v||^2 x = 0,$$

so $v = 0$. This shows that, picking a basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, we obtain n vector fields on S^N $V_1, \ldots, V_n$, linearly independent at each $x \in S^N$.

Corollary. (See [L-M, p. 457]; follows from the representation theory of Cl_n) Write $N + 1 = 2^{4a+b}(2t + 1)$, with $0 \leq b \leq 3$. Then the largest n such that $\mathbb{R}^{N+1}$ is a Cl_n-module is $n = 8a + 2^b - 1$ (in particular $n = 0$ if N is even, since then $a = b = 0$).

Remark. It is a famous theorem of J.F. Adams (1962) that this n gives indeed the rank of S^n (largest number of vector fields linearly independent at each point). For instance, $\mathrm{rank}(S^{\text{even}}) = 0$, as expected. We confirm $\mathrm{rank}(S^1) = 1, \mathrm{rank}(S^3) = 3$, $\mathrm{rank}(S^7) = 7$ (the only parallelizable spheres), and $\mathrm{rank}(S^5) = 1$. (It is well known, and follows from the embedding of S^{2k+1} as the unit sphere in $\mathbb{C}^{k+1}$, that $\mathrm{rank}(S^{\text{odd}}) \geq 1$: every odd-dimensional sphere admits a non-vanishing vector field.)

1.9. **Review of the Associated Vector Bundle Construction.** Given any principal G-bundle $\pi : P \to X$ (a manifold), where G is a Lie group, and a representation $\rho : G \to \mathrm{GL}(W)$ (invertible linear transformations of a vector space W), we construct a vector bundle over X with typical fiber W, the associated bundle $P \times_\rho W$, as follows. Recall P admits a free right G-action, transitive on fibers. Then define a left G-action on the product $P \times W$ via:

$$g \cdot (e, w) = (e \cdot g^{-1}, \rho(g)w).$$

This action is free (since the action of G on P is), with disjoint orbits, and the quotient space $P \times_\rho W$ (the orbit space $(P \times W)/G$ of this left G-action) is a well-defined locally trivial vector bundle over X, with typical fiber W. In what follows we describe the local trivializations and transition maps of this vector bundle, in terms of those of P.

Let $\hat{\pi} : P \times W \to P \times_\rho W$ be the quotient projection. This is also the quotient projection of the equivalence relation on $P \times W$ ("orbit equivalence"):

$$(e, w) \sim (eg^{-1}, \rho(g)w) \quad \text{if } e' = eg^{-1} \text{ and } w' = \rho(g)w.$$

Denote the local trivializations (with respect to an open cover $\mathcal{U} = \{U_\alpha\}$ of X) of the principal G-bundle P by:

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \quad \text{(diffeomorphisms)},$$
$$\psi_\alpha(e) = (\pi(e), \psi_\alpha^G(e)), \quad \psi_\alpha^G : P|_{U_\alpha} \to G,$$

satisfying equivariance: $\psi_\alpha^G(e \cdot g) = \psi_\alpha^G(e)g$, $\forall g \in G, e \in P|_{U_\alpha}$.

The transition diffeomorphisms $\psi_\alpha \circ \psi_\beta^{-1} : U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G$ have the form:

$$\psi_\alpha \circ \psi_\beta^{-1}(x, g) = (x, \varphi_{\alpha\beta}(x)g), \quad \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G \,\text{(smooth)}.$$

Also $P \times_\rho W$ is a locally trivial vector bundle, with projection map:

$$p : P \times_\rho W \to X, \quad p[e, w] = \pi(e).$$

(Clearly well-defined on the equivalence class $[e, w]$ of $(e, w) \in P \times W$). Its local trivializations are given by:

$$\hat{\psi}_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times W, \quad [e, w] \mapsto (\pi(e), \rho(\psi_\alpha^G(e))w).$$

It follows from the equivariance property of the ψ_α^G that this is well defined, since if $(e, w) \sim (e', w')$:

$$\rho(\psi_\alpha^G(e'))w' = \rho(\psi_\alpha^G(eg^{-1}))\rho(g)w = \rho(\psi_\alpha^G(e)g^{-1})\rho(g)w = \rho(\psi_\alpha^G(e))w,$$

using the fact ρ is a group homomorphism.

The following very natural lemma needs proof. Define transition maps for $P \times_\rho W$:

$$\hat{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(W) \quad \text{by} \quad \hat{\psi}_\alpha \circ \hat{\psi}_\beta^{-1}(x, w) = (x, \hat{\varphi}_{\alpha\beta}(x)w), x \in U_\alpha \cap U_\beta$$

Lemma. $\hat{\varphi}_{\alpha\beta} = \rho \circ \varphi_{\alpha\beta}$.

Proof.[1]

Let $\hat{\psi}_\beta^{-1}(x, w) = [e_\beta, w_\beta], \quad x = \pi(e_\beta) \in U_\alpha \cap U_\beta, w_\beta \in W.$

Thus $(x, w) = \hat{\psi}_\beta([e_\beta, w_\beta]) = (x, \rho(\psi_\beta^G(e_\beta))w_\beta)$, so:

$$\rho(\psi_\beta^G(e_\beta))w_\beta = w \qquad (1)$$

We need to show $\hat{\psi}_\alpha([e_\beta, w_\beta]) = (x, \rho(\varphi_{\alpha\beta}(x))w)$, or:

$$\rho(\psi_\alpha^G(e_\beta))w_\beta = \rho(\varphi_{\alpha\beta}(x))w.$$

Now, referring to the trivializations of P:

$$\psi_\beta(e_\beta) = (x, \psi_\beta^G(e_\beta)), \quad \text{or} \quad e_\beta = \psi_\beta^{-1}(x, g_\beta), \quad g_\beta = \psi_\beta^G(e_\beta).$$

Then:

$$\psi_\alpha^G(e_\beta) = \varphi_{\alpha\beta}(x)\psi_\beta^G(e_\beta) = \varphi_{\alpha\beta}(x)g_\beta \qquad (2)$$

[1] It is probably more fun to do it as an exercise than to read the proof.

Thus:
$$\rho(\psi_\alpha^G(e_\beta))w_\beta = \rho(\varphi_{\alpha\beta}(x))\rho(\psi_\beta^G(e_\beta))w_\beta = \rho(\varphi_{\alpha\beta}(x))w,$$
as we wished to show. (The first equality follows from (2), the second from (1).)

1.10. **The Clifford Algebra Bundle of a Riemannian Manifold** (X, g)**.** As seen in §4 above, $G = O_n$ (or SO_n) acts on Cl_n by algebra homomorphisms (in several ways), in particular by invertible linear transformations of Cl_n (regarded as a real vector space). Any of these actions leads to an associated vector bundle over X. It turns out that, if we want this vector bundle to have a Clifford algebra structure (at each $x \in X$), we need to use the adjoint representation $Ad : O_n \to GL(\mathrm{Cl}_n), Ad(A)\psi = A\psi A^t$. Thus we consider:
$$\mathrm{Cl}(X) = F_X \times_{Ad} \mathrm{Cl}_n,$$
where F_X is the orthonormal frame bundle of X (or orthonormal positive frame bundle if X is oriented). That is, the orbit equivalence relation is:
$$(e, \psi) \sim (eg^{-1}, Ad(g)\psi); \quad e \in F_X, g \in O_n, \psi \in \mathrm{CL}_n$$

To see that this is not just a vector bundle, but a bundle over X of Clifford algebras, isomorphic to Cl_n at each point $x \in X$, define the Clifford product on $\mathrm{Cl}(X)$ by:
$$[e, \varphi] \cdot [e', \psi] = [e, \varphi \cdot Ad(g^{-1})\psi] = [e, \varphi \cdot g^{-1}\psi g]$$
where $e, e' = eg^{-1} \in F_X, \varphi, \psi \in \mathrm{Cl}(\mathbb{R}^n), g \in O_n$.

(The idea is to first refer ψ to the same frame e, using $(e', \psi) \sim (e'g, Ad(g^{-1})\psi) = (e, Ad(g^{-1})\psi)$) To see this is well-defined, we consider different representatives of the equivalence classes:
$$(e, \varphi) \sim (eA^{-1}, A\varphi A^{-1}), \quad (e', \psi) \sim (e'B^{-1}, B\psi B^{-1}), \quad A, B \in O_n, \varphi, \psi \in \mathrm{Cl}_n.$$
Note: $e'B^{-1} = eg^{-1}B^{-1} = eA^{-1}Ag^{-1}B^{-1} = eA^{-1}(BgA^{-1})^{-1}$, so we must replace g by BgA^{-1} when computing the product:
$$[eA^{-1}, A\varphi A^{-1}]\cdot[e'B^{-1}, B\psi B^{-1}] = [eA^{-1}, A\varphi A^{-1}\cdot Ad(BgA^{-1})^{-1}(B\psi B^{-1})] = [eA^{-1}, A(\varphi\cdot g^{-1}\psi g)A^{-1}]$$
and we see that $(e, \varphi \cdot g^{-1}\psi g) \sim (eA^{-1}, Ad(A)(\varphi \cdot g^{-1}\psi g))$, as desired.

Intuitively, one thinks of a section $\varphi \in \Gamma(\mathrm{Cl}_n(X))$, locally, as a pair (e, φ_0), where $e = (e_i)$ is a local orthonormal frame and $\varphi_0 \in \mathrm{Cl}(\mathbb{R}^n)$, modulo the equivalence relation defined by the adjoint action of O_n.

Exercise. (Instructive). Show this wouldn't work if we tried to define $\mathrm{Cl}(X)$ using the standard left representation of O_n on Cl_n.

Example. (Connection with elementary linear algebra.) In Linear Algebra, we compute the expression of $v \in \mathbb{R}^n$ in two different (say orthonormal) bases e, e' as follows:
$$v = \sum_j x_j e_j = \sum_i y_i e'_i \text{ where } e_j = \sum_i e'_i g_{ij}$$
$$\sum_j x_j e_j = \sum_{i,j} x_j e'_i g_{ij} = \sum_i (\sum_j g_{ij} x_j) e'_i \Rightarrow y_i = \sum_j g_{ij} x_j,$$
which we may write as:
$$e' = eG^{-1} \Rightarrow [v]_{e'} = G[v]_e \quad \text{(matrices acting on `column vectors')}.$$

This suggests the equivalence relation of pairs (frame, coordinate) representing the same vector:

$$(e, x) \sim (eG^{-1}, Gx).$$

In terms of associated vector bundles to the orthonormal frame bundle F_X of a Riemannian manifold X, this amounts to the isomorphism:

$$F_X \times_{\rho_n} \mathbb{R}^n \approx TX,$$

where $\rho_n : O_n \to \mathcal{L}(\mathbb{R}^n)$ is the standard action (matrices acting on 'column vectors' on the left.)

Now recall how the matrix expressions of a linear transformation behave under a change of basis. The matrix $[T]_e$ of a linear transformation $T \in \mathcal{L}(\mathbb{R}^n)$ in the basis e is defined by:

$$[Tv]_e = [T]_e[v]_e.$$

And from the above, if $e' = eG^{-1}$ is a second basis, we have:

$$[Tv]_{e'} = G[Tv]_e = G[T]_e[v]_e = G[T]_e G^{-1}[v]_{e'},$$

proving the classical formula: $[T]_{e'} = G[T]_e G^{-1}$.

In other words, in terms of an equivalence relation on pairs (frame, square matrix) representing the same linear transformation, this says:

$$(e, A) \sim (eG^{-1}, Ad(G)A).$$

Thus, matrices transform via the adjoint action, under a change of basis. For this reason, the algebra of fiber-wise linear transformations of the tangent bundle TX of a Riemannian manifold X can be constructed as the bundle associated to the frame bundle F_X by the adjoint representation of O_n on the space of real $n \times n$ matrices:

$$\mathcal{L}(TX) \approx F_X \times_{Ad} \mathbb{M}_{n\times n}.$$

That this bundle has a well-defined algebra structure (under composition) is shown just as above for the case of $\mathrm{Cl}(X)$.