COMPATIBLE CONNECTIONS AND CURVATURE ON Cl(X)-MODULES

Let W be a Cl(X)-module: a complex vector bundle W (of $\mathbb{C}$-fiber dimension N) with hermitian metric, endowed, at each point $x \in X$, with a Clifford multiplication $Cl(T_xX) \to \mathcal{L}_{\mathbb{C}}(W_x)$ depending smoothly on x. (Recall this Cliford action must itself satisfy a local triviality condition.) We denote by F_X the orthonormal frame bundle of X, a principal SO_n bundle.

Let $(s_a(x)), (e_i(x))$ be suitable local orthonormal frames for $W_{|U}, F_{X|U}$. The frames determine matrices of connection 1-forms for the Levi-Civita connection $\nabla^g = \nabla^{LC}$ on (X, g) and the compatible connection D on the Cl(X, g)-module W, $\omega^g = \omega^{LC} \in T^*X_{|U} \otimes \mathfrak{so}_n, \omega \in T^*X_{|U} \otimes \mathfrak{u}_N$:

$$\nabla^g_v e_i = \sum_j \omega_{ji}(v) e_j, \quad D_v s_a = \sum_b \omega_{ba}(v) s_b,$$

equivalently:

$$\omega^g_{ji}(v) = \langle \nabla^g_v e_i, e_j \rangle, \quad \omega_{ba} = \langle D_v s_a, s_b \rangle.$$

(Note the order of indices, the usual one for matrix elements when matrices multiply 'column vectors' on the left).

Note also the matrices ω are complex-valued, and skew-hermitian: $\omega^\dagger = -\omega$. We also represent (locally) the Clifford action of e_i on W, in the frame (s_a), by a skew-hermitian matrix $C(e_i) \in \mathfrak{u}_N$, with entries defined by:

$$C(e_i) s_a = \sum_b c_{ba}(e_i) s_b, \quad \text{or } c_{ba}(e_i) = \langle C(e_i) s_a, s_b \rangle.$$

('Suitable frames' means, by definition, the matrix entries $c_{ab}(e_i)(x)$ are constant functions on U.)

Now compute, using the conditions defining compatibility:

$$\begin{aligned} e_k(c_{ba}(e_i)) &= \langle D_{e_k}(c(e_i)s_a), s_b \rangle + \langle C(e_i)s_a, D_{e_k} s_b \rangle \\ &= \langle c(\nabla^g_{e_k} e_i) s_a, s_b \rangle + \langle c(e_i)(D_{e_k} s_a), s_b \rangle + \langle C(e_i) s_a, D_{e_k} s_b \rangle \\ &= \langle \sum_j \omega^g_{ji}(e_k) C(e_j) s_a, s_b \rangle + \langle C(e_i) \sum_d \omega_{da}(e_k) s_d, s_b \rangle + \langle C(e_i) s_a, \sum_d \omega_{db}(e_k) s_d \rangle \\ &= \sum_i \omega^g_{ji} c_{ba}(e_j) + \sum_d \omega_{da}(e_k) c_{bd}(e_i) + \sum_d \bar{\omega}_{db}(e_k) c_{da}(e_i) \\ &= \sum_j \omega^g_{ji}(e_k) c_{ba}(e_j) + (c(e_i)\omega(e_k))_{ba} - (\omega(e_k) c(e_i))_{ba} \\ &= \sum_j \omega^g_{ji}(e_k) c_{ba}(e_j) - [\omega(e_k), c(e_i)]_{ba} \end{aligned}$$

Since the matrix coefficients $c_{ba}(e_i)$ are constant over U for suitable frames, we conclude the connection matrices for a compatible connection must satisfy the relation:

$$\sum_j \omega^g_{ij}(e_k) C(e_j) + [\omega(e_k), C(e_i)] = 0, \quad \text{for all } i, k.$$

We claim that a *solution* to this system is given by the matrices:

$$\omega^s(e_k) = \frac{1}{4}\sum_{l,m} \omega^g_{ml}(e_k)C(e_l)C(e_m).$$

First, an easy calculation verifies these matrices are indeed skew-hermitian (exercise.)

Now compute, for $i \neq k$:

$$\omega^s(e_k)C(e_i) - C(e_i)\omega^s(e_k) = \frac{1}{4}\sum_{l,m} \omega^g_{ml}(e_k)[C(e_l)C(e_m)C(e_i) - C(e_i)C(e_l)C(e_m)]$$

$$= \frac{1}{4}\sum_{l,m} \omega^g_{ml}(e_k)[-C(e_l)C(e_i)C(e_m) - 2\delta_{im}C(e_l) - (-C(e_l)C(e_i)C(e_m) - 2\delta_{il}C(e_m))]$$

$$= \frac{1}{2}\sum_{l,m} \omega^g_{ml}(e_k)[-\delta_{im}C(e_l) + \delta_{il}C(e_m)]$$

$$= \frac{1}{2}(-\sum_{l} \omega^g_{il}(e_k)C(e_l) + \sum_{m} \omega^g_{mi}(e_k)C(e_m)) = \sum_{i,l} \omega^g_{li}(e_k)C(e_l)$$

Thus we see that, for all i, k:

$$[\omega^s(e_k), C(e_i)] = \sum_{ij} \omega^g_{ij}(e_k)C(e_i).$$

And therefore we indeed have:

$$\sum_j \omega^g_{ij}(e_k)C(e_j) + [\omega(e_k), C(e_i)] = \sum_j (\omega^g_{ij}(e_k) + \omega^g_{ji}(e_k))C(e_j) = 0.$$

Remark/exercise. Recall (from Riemannian geometry) that connection 1-forms ω_{ij} associated to local frames (as well as the associated curvature 2-forms Ω_{ij}) must satisfy a compatibility condition under change of frames, if they are to define connections on Riemannian vector bundles. For the Levi-Civita connection on TX:

$$e_i = \sum_j e'_j g_{ji}, \quad g(x) = (g_{ij}(x)) \in SO_n \Rightarrow \omega^{LC} = g^t {\omega'}^{LC} g + g^t dg, \quad \Omega^{LC} = g^t {\Omega'}^{LC} g.$$

And similarly for the hermitian complex vector bundle W:

$$s_a = \sum_b \tilde{s}_b u_{ba}, \quad U(x) = (u_{ab}(x)) \in U_N \Rightarrow \omega = U^\dagger \tilde{\omega} U + U^\dagger dU, \quad \Omega = U^\dagger \tilde{\Omega} U.$$

So, strictly speaking, we need to verify that the connection 1-forms ω^s defined above satisfy this compatibility condition under change of frames, given that the ω^{LC} do.

Perhaps this follows directly from the condition found at the outset, connecting the ω^{LC} and the ω^s. Also, observe that if both pairs of frames (e, s) and $(e', \tilde{s})$ are to be 'suitable' (as defined earlier), there should be a condition connecting the change of frame maps g and U. Can you find it?

Perhaps it is (check!):

$$dU(e_k)C(e_l)U^\dagger + UC(e_l)dU^\dagger(e_k) + \sum_i (g^t dg)_{li}(e_k)UC(e_i)U^\dagger = 0 \quad \forall k, l.$$

(There may be a geometric way to understand this constraint.)

Exercise. Use the constraint found above to establish that if the connection 1-forms $\omega = (\omega_{ab}) \in \Lambda^1_U \otimes \mathfrak{u}_N$ (referring to suitable frames $(e_i), (s_a)$ for TX and W) satisfy the equation for compatibility of the connection, then changing to another set $(e'_i), (\tilde{s}_a)$ of suitable frames (as described above) yields new connection 1-forms $\tilde{\omega}$ which still satisfy the compatibility condition (with respect to the Levi-Civita connection on $\mathrm{Cl}(X)$, corresponding to 1-forms $\omega^{LC}_{ij}, {\omega'}^{LC}_{ij}$ in the frames $(e_i), (e'_i)$ respectively.)