The Inner Product on $\mathrm{Cl}(V, g)$ and the Adjoint Action

(a) The inner product g on V induces an inner product on $\Lambda^*(V)$, defined by declaring $\{e_I\}_{\mathcal{I}_p}$ as an orthonormal basis of $\Lambda^p(V)$ if (e_i) is an orthonormal basis of V, and $\Lambda^p(V) \perp \Lambda^q(V)$ for $p \neq q$. Via the vector space isomorphism $\mathrm{Cl}(V) \cong \Lambda^*(V)$, this induces an inner product on $\mathrm{Cl}(V)$.

Unit vectors $e \in V$ act as isometries of $\mathrm{Cl}(V)$ under Clifford multiplication:

$$\langle e \cdot \phi, e \cdot \psi \rangle = \langle \phi, \psi \rangle \quad \text{if } e \in V,\ |e|^2 = 1 \quad (\text{say } \phi, \psi \in \Lambda^p(V))$$

To see this, recall $e \cdot \phi$ corresponds to $e \wedge \phi - i_e \phi \in \Lambda^{p+1} + \Lambda^{p-1}$. Thus:

$$\langle e \cdot \phi, e \cdot \psi \rangle = \langle e \wedge \phi, e \wedge \psi \rangle + \langle i_e \phi, i_e \psi \rangle$$

$$= \langle \phi, i_e(e \wedge \psi) \rangle + \langle \psi, e \wedge (i_e \phi) \rangle = \langle \phi, \psi \rangle$$

(b) The same is true under right Clifford multiplication:

$$\langle \phi \cdot e, \psi \cdot e \rangle = \langle \phi, \psi \rangle \quad \text{if } e \in V,\ |e|^2 = 1 \quad (\text{say } \phi, \psi \in \Lambda^p(V)).$$

Just recall $e \cdot \psi$ corresponds to $(-1)^k (e \wedge \psi + i_e \psi)$ if $\psi \in \Lambda^p$. Thus:

$$\langle \phi \cdot e, \psi \cdot e \rangle = \langle e \wedge \phi, e \wedge \psi \rangle + \langle i_e \phi, i_e \psi \rangle = \langle \phi, \psi \rangle$$

(c)Thus, the inner product $\langle \cdot, \cdot \rangle$ is also preserved by the adjoint action. Recall the definition:

Let $\mathrm{Cl}^\times(V)$ be the group of units in $\mathrm{Cl}(V)$ — elements admitting two-sided inverses (e.g., any nonzero $v \in V$, since $v^{-1} = -\frac{v}{|v|^2}$).

Define:

$$\mathrm{Ad}(\psi)\phi = \psi \cdot \phi \cdot \psi^{-1} \quad (\psi \in \mathrm{Cl}(V),\ \phi \in \mathrm{Cl}^*(V)).$$

Then if $e \in V$, $|e| = 1$, we have:

$$\langle \mathrm{Ad}(e)\phi, \mathrm{Ad}(e)\psi \rangle = \langle e\phi e^{-1}, e\psi e^{-1} \rangle = \langle \phi, \psi \rangle,$$

using (a) and (b) above.

Iterating, this is also true for the subgroup $\mathrm{Pin}(V) \subset \mathrm{Cl}^{\times}(V)$ generated by unit vectors $e \in V$ under Clifford multiplication.

$$\langle \mathrm{Ad}(\psi)\phi_1, \mathrm{Ad}(\psi)\phi_2 \rangle = \langle \phi_1, \phi_2 \rangle \quad \forall \phi_1, \phi_2 \in \mathrm{Cl}(V), \psi \in Pin(V).$$

(d) Consider a smooth curve $\psi(t)$ in $\mathrm{Cl}^{\times}(V)$ such that $\psi(0) = 1$. Assume $Ad(\psi(t))$ acts as an isometry. Differentiating:

$$\langle Ad(\psi(t)\phi, Ad(\psi(t)\zeta \rangle \equiv \langle \phi, \zeta \rangle \quad \forall t,$$

we find:

$$0 = \frac{d}{dt}_{|t=0} \langle \psi(t) \cdot \phi \cdot \psi(t)^{-1}, \psi(t) \cdot \zeta \cdot \psi(t)^{-1} \rangle = \langle \psi'(0) \cdot \phi - \phi \cdot \psi'(0), \zeta \rangle + \langle \phi, \psi'(0) \cdot \zeta - \zeta \cdot \dot{\psi}'(0) \rangle$$

$$= \langle [\psi'(0), \phi], \zeta \rangle + \langle \phi, [\psi'(0), \zeta] \rangle$$

The 'little ad map' ad_ψ is defined as the derivation of $\mathrm{Cl}(V)$ given by the algebra commutator:

$$ad_\psi \phi = [\psi, \phi] = \psi \cdot \phi - \phi \cdot \psi \text{ for } \psi, \phi \in \mathrm{Cl}(V).$$

It is easy to show from the definition that for $\phi \in \mathrm{Cl}(V)$, ad_ϕ acts on $\mathrm{Cl}(V)$ by algebra derivations:

$$\mathrm{ad}_\phi(\psi_1 \cdot \psi_2) = (\mathrm{ad}_\phi \psi_1) \cdot \psi_2 + \psi_1 \cdot (\mathrm{ad}_\phi \psi_2)$$

We conclude that if $\psi(t)$ is an isometry for all t and $\psi(0) = 1$, then $\mathrm{ad}(\psi'(0))$ is skew-adjoint:

$$\langle ad_{\psi'(0)} \phi, \zeta \rangle + \langle \phi, ad_{\psi'(0)} \zeta \rangle = 0$$

In particular, this is true if $\psi(t) \in \mathrm{Pin}(V)$ for all t and $\psi(0) = 1$.

Example: For (e_i) an orthonormal basis of V and $i \neq j$, consider the curve:

$$\psi(t) = (\cos t\, e_i + \sin t\, e_j) \cdot (-\cos t\, e_i + \sin t\, e_j)$$

$\Psi(t) \in \mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^{+}(V)$, since it is the Clifford product of two unit vectors; $\psi(0) = 1$ and $\psi'(0) = e_i \cdot (-e_j) + e_j \cdot e_i = 2 \cdot e_i \cdot e_j$. Thus, $\mathrm{ad}_{e_i e_j}$ is skew-adjoint on $\mathrm{Cl}(V)$.

(e) **Lemma.** Let $v \in V \subset \mathrm{Cl}(V)$, $v \neq 0$ (assume $\|v\| = 1$, so $v^{-1} = -v$). Then $\mathrm{Ad}_v(V) = V$. In fact:

$$-\mathrm{Ad}_v(w) = w - 2\langle v, w \rangle v = R_v(w), \quad \forall w \in V$$

($R_v(w)$ is the reflection of w on the hyperplane $v^\perp$)

Proof:

$$-\mathrm{Ad}_v(w) = -v \cdot w \cdot v^{-1} = v \cdot w \cdot v = -v \cdot v \cdot w - 2\langle v, w \rangle v = w - 2\langle v, w \rangle v$$

(Note that, indeed, $R_v \in O(V)$).

Remark: In contrast, ad_v does not preserve V if $v \neq 0$: If $v \perp w$,

$$\mathrm{ad}_v(w) = v \cdot w - w \cdot v = 2v \cdot w,$$

which corresponds to $2(v \wedge w - \langle v, w\rangle) = 2v \wedge w \in \Lambda^2(V)$.

(f) Denote by $\mathfrak{so}(V)$ the space of skew-adjoint transformations of V, that is, the Lie algebra of $SO(V)$. We have the important isomoprhism:

$$\Lambda^2 V \cong \mathfrak{so}(V),$$

made explicit by assigning to $u \wedge v \in \Lambda^2 V$ the skew-symmetric endomorphism of V:

$$(u \wedge v)w = \langle u, w\rangle v - \langle v, w\rangle u.$$

(Note this maps u to v and v to $-u$, if u, v are orthogonal unit vectors in V.)

(g) *Example/exercise:* Consider $u(t), v(t)$ curves on the unit sphere of V: $\|u(t)\| = \|v(t)\| = 1$ for all t. Assume $u(0) = v(0) = p$. Let:

$$\psi(t) = \mathrm{Ad}_{v(t)} \circ \mathrm{Ad}_{(u(t))} = R_{v(t)} \circ R_{u(t)},$$

a composition of two reflections, hence in $SO(V)$. Note $\psi(0) = Id_V$, so $\psi'(0) \in \mathrm{Skewsymm}(V) = \mathfrak{so}(V)$, the Lie algebra of $SO(V)$. Prove that:

$$\psi'(0)w = (p \wedge (u'(0) - v'(0)))w.$$

(h) In the example in (d) above, we saw that if (e_i) is an orthonormal basis of V and $i \neq j$, then $ad_{e_i e_j}$ is an element of $\mathfrak{so}(V)$. The following important lemma identifies the element of $\Lambda^2 V$ corresponding to it under the isomorphism described in (f):

Lemma. We have:

$$\mathrm{ad}_{e_i e_j}(v) = 2(e_i \wedge e_j),$$

as elements of $\mathfrak{so}(V)$. (In particular, $ad_{e_i e_j}$ maps V to V.)

Proof: Let $v \in V$.

$$\mathrm{ad}_{e_i e_j}(v) = e_i \cdot e_j \cdot v - v \cdot e_i \cdot e_j$$

$$= -e_i \cdot v \cdot e_j - 2\langle v, e_j\rangle e_i - (-e_i \cdot v \cdot e_j - 2\langle v, e_i\rangle e_j)$$

$$= 2\left(\langle v, e_i\rangle e_j - \langle v, e_j\rangle e_i\right) = 2(e_i \wedge e_j)(v)$$

Action of $ad_{e_ie_j}$ on $\Lambda^p(V)$ ((e_i) orthonormal basis of V).

Since $\mathrm{ad}_{e_ie_j}$ preserves V and acts as a derivation of $\mathrm{Cl}(V)$, it also preserves $\Lambda^p V$, for each p:

$$\mathrm{ad}_{e_ie_j}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_k e_{i_1} \wedge \cdots \wedge \mathrm{ad}_{e_i \wedge e_j}(e_{i_k}) \wedge \cdots \wedge e_{i_k}$$

Note that $e_i \wedge e_j \in \mathfrak{so}(V)$ takes $e_i \mapsto e_j$, $e_j \mapsto -e_i$, and all other $e_k \mapsto 0$.

Thus, $\mathrm{ad}_{e_ie_j}(e_I)$ is nonzero only if exactly one of i, j is in the multi-index $I \in \mathcal{I}_p$.

Using this, it's not hard to show (exercise):

Lemma: If $\mathrm{ad}_{e_ie_j}(\phi) = 0$ for all i, j (where $\phi \in \Lambda^p V$ with $p = 1, 2, \ldots, n-1$), then $\phi = 0$.

(The dimension restriction on ϕ is necessary.

Note, for instance, $\mathrm{ad}_{e_ie_j}(e_1 \wedge \cdots \wedge e_n) = 0$ for all i, j).