Topology Pres 1: Rotmann

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Lemma 0.1. *Let $y_1, y_2 \in Y$ be such that $y_1 \sim y_2$. Then $y_1 = y_2$.*

Proof. Note that if $y_1 \in f(A)$, then there is some $a \in A$ such that $y_1 = f(a)$. Thus

$$y_2 \sim y_1 \sim a.$$

But the only element of Y that a shares an equivalence class with is $f(a)$. Thus $y_2 = f(a) = y_1$ and so we may assume $y_1, y_2 \notin f(A)$. But then $[y_i] = \{y_i\}$ so if $y_1 \sim y_2$ it must be the case that $y_1 = y_2$. □

Lemma 0.2. *Let x_1, x_2 be distinct elements of X. Then $x_1 \sim x_2$ if and only if $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$.*

Proof. The reverse direction is clear as this means x_1, x_2 share an equivalence class with some $y = f(x_i)$, thus share an equivalence class with one another. The forward direction we prove by contrapositive. Assume x_1 or $x_2 \notin A$ or they map to distinct y_1, y_2. By the above lemma, if $f(x_1) = y_1 \neq y_2 = f(x_2)$, then $y_1 \not\sim y_2$ and thus $x_1 \not\sim x_2$. Thus we may assume without loss of generality that $x_1 \notin A$. But then $[x_1] = \{x_1\}$, and thus does not contain x_2. Thus $x_1 \not\sim x_2$. Thus if it is the case $x_1 \sim x_2$ it must be the case that $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. □

8.12) Let X, Y be spaces and $A \subset X$ an nonempty closed subset. Let $f : A \to Y$ be continuous and $\nu : X \sqcup Y \to X \sqcup_f Y$ the natural map.

i) Assume $C \subset X \sqcup Y$ is such that $C \cap X$ is closed in X. Show $\nu(C)$ is closed in $X \sqcup_f Y$ iff $(C \cap Y) \cup f(C \cap A)$ is closed in Y.

Proof. $\implies$:

To begin, assume that $\nu(C) \subset X \sqcup_f Y$ is closed. Then since ν is a continuous map, we have $\nu^{-1}(\nu(C))$ is a closed subset of $X \sqcup Y$. For notational convenience, we now adopt the following notation

$$C_X := C \cap X \qquad C_Y := C \cap Y \qquad C_A := C \cap A$$

We now show that this closed subset is equal to

$$C \cup f(C_A) \cup f^{-1}f(C_A) \cup f^{-1}(C_Y).$$

We show the forward containment in cases:

- If $t \in C$, then $\nu(t) \in \nu(C)$.
- If $t \in f(C_A)$, then there is some $s \in C_A \subset C$ such that $f(s) = t$. Since $s \sim t$ and $\nu(s) \in \nu(C)$ we have $\nu(t) \in \nu(C)$.
- If $t \in f^{-1}f(C_A)$, then $f(t) = r \in f(C_A)$. By the above example, $\nu(r) \in \nu(C)$. Since $t \sim r$ and $\nu(r) \in \nu(C)$ we have $\nu(t) \in \nu(C)$.

– If $t \in f^{-1}(C_Y)$, then $f(t) = r \in C_Y \subset C$. By the above example, $\nu(r) \in \nu(C)$. Since $t \sim r$ and $\nu(r) \in \nu(C)$ we have $\nu(t) \in \nu(C)$.

Thus
$$\nu^{-1}\nu(C) \supset C \cup f(C_A) \cup f^{-1}f(C_A) \cup f^{-1}(C_Y).$$

To see the reverse inclusion, let $s \in \nu^{-1}\nu(C) \subset X \sqcup Y$. Note if $s \in C$, then we are done so we may assume not. Thus we have $s \in (X \setminus C_X) \sqcup (Y \setminus C_Y)$. Since $\nu(s) \in \nu(C)$, there is some $t \in C$ such that $\nu(s) = \nu(t)$. Thus $t \in C_X \sqcup C_Y$.

– Assume $s \in (X \setminus C_X)$ and $t \in C_X$. Note that if $x, x' \in X$, then $x \sim x'$ iff $x, x' \in A$ and $f(x) = f(x')$ as the relation $\sim$ associates elements of A with elements Y and does not (directly) associate elements of X with other elements of X (relations on elements of X only come from transitivity). Thus $s, t \in A$, and so $t \in C_A$. As such $f(t) = r \in f(C_A)$. Note that we now have $s \sim t \sim r$. Since we have $s \sim t$, we must have $f(s) = f(t) = r \implies s \in f^{-1}f(C_A)$

– Assume $s \in X \setminus C_X$ and $t \in C_Y$. Then there is some $r \in f^{-1}(C_Y) \subset X$ such that $f(r) = t$. Thus we again have $s \sim t \sim r$ so $s, r \in A$ and thus it must be the case that $r \in C_A$ and $f(s) = f(r) = t \in C_Y$ so we see $s \in f^{-1}(C_Y)$.

– Assume $s \in Y \setminus C_Y$. If $t \in C_X$, then it must be the case that $f(t) = s$ (as this is the only relation between elements of X and Y) which in turn would mean $t \in A$ and thus $t \in C_A \implies s \in f(C_A)$. Thus we may assume $t \in C_Y$. We claim this will give a contradiction. To this end, note that as with elements of X, any relation between elements of Y is deduced from transitivity. Thus elements $y, y' \in Y$ are such that $y \sim y'$ iff $y = y'$. Since we assume that $s \in Y \setminus C_Y$ and $t \in C_Y$, this obviously cannot occur.

We have now shown that $s \in C \cup f(C_A) \cup f^{-1}f(C_A) \cup f^{-1}(C_Y)$ thus establishing the equality of the above sets. Since they are closed, it follows that
$$Y \cap \nu^{-1}\nu(C) = (Y \cap C) \cup f(C_A)$$
is a closed subset of Y. Since $C_A := C \cap A$, this gives us the desired result.

$\Longleftarrow$:

Note that
$$\nu^{-1}\nu(C) = C \cup f(C_A) \cup f^{-1}f(C_A) \cup f^{-1}(C_Y)$$
can also be written as
$$(C_X) \cup (C_Y) \cup f(C_A) \cup f^{-1}\left(f(C_A) \cup C_Y\right).$$
By assumption we have $f(C_A) \cup C_Y$ is a closed subset of Y and C_X is a closed subset of X. Further, since f is continuous and A is closed $f^{-1}\left(f(C_A) \cup C_Y\right)$ (which is closed in A) is closed in X. Thus we have a finite union of closed sets, thus it is closed. □

ii) Show the composite
$$Y \hookrightarrow X \sqcup Y \to X \sqcup_f Y$$
is a homeomorphism from Y to a subspace of $X \sqcup_f Y$.

Proof. Since the composition map is continuous (being composition of continuous maps) and bijective, we show it is a closed map and conclude it is a homeomorphism. Let $C \subset Y$ be a closed set. Then the inclusion of C into $X \sqcup Y$, call it $\hat{C}$ is a closed subset of $X \sqcup Y$ satisfying $(\hat{C} \cap Y) \cup f(\hat{C} \cap A)$ is closed in Y, thus $\nu(\hat{C})$ is a closed subset of $X \sqcup_f Y$. But $\nu(\hat{C}) = \nu \circ \iota(C)$, thus the composition is a closed map and so it is a homeomorphism. □

iii) Show the composite

$$\Phi : X \hookrightarrow X \sqcup Y \to X \sqcup_f Y$$

maps $X \setminus A$ homeomorphically onto an open subset of $X \sqcup_f Y$.

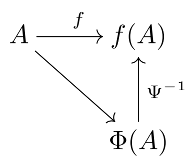
Proof. To begin, we see the composite map is a bijection by the second lemma above. We thus show it is an open map to conclude it is a homeomorphism. To this end, let $U \subset X \setminus A$ be an open set. Let $\hat{U}$ be the image of U be the inclusion of U in $X \sqcup Y$, which we see is an open subset. Observe that it is an open set and that

$$\nu(\hat{U}) = \{[x] : x \in \hat{U}\} \implies \nu^{-1}\nu(\hat{U}) = \hat{U}$$

But since $\hat{U}$ is an open subset of $X \sqcup Y$, $\nu(\hat{U})$ is an open subset in $X \sqcup_f Y$, as the open subset are exactly those whose preimage is an open subset of $X \sqcup_f Y$. Since the composition is clearly continuous (being composition of continuous maps) and a bijective open map, it is a homeomorphism. □

iv) Under the identification map in *(ii)*, show one may regard $\Phi|_A$ as the attaching map f.

Proof. Firstly, let Ψ be the composition map from *(ii)*. We show the following diagram commutes



To this end, let $a \in A$ and $s = f(a)$. Then $a \sim s$ so $\Phi(a) = \nu(a) = \nu(s) = \Psi(s)$. Thus $f(a) = s = \Psi^{-1} \circ \Phi(a)$ so the diagram commutes and we may regard f as being equal to $\Psi^{-1} \circ \Phi(a)$. But since Ψ is a homeomorphism that identifies s and $[s]$, it makes just as much sense to consider f as just being equal to $\Phi|_A$, as the value of $\Phi(a)$ uniquely determines the value of $\Psi^{-1} \circ \Phi(a)$, thus determines f. □

8.13) Suppose that in exercise 8.12 that A is compact and X, Y are Hausdorff.

i) Show that the natural map ν is a closed map.

Proof. Let $C \subset X \sqcup Y$ be a closed set. Then $C \cap A$ is a closed subset of A, a compact set, thus is compact. Thus $f(C \cap A)$ is a compact subset of a Hausdorff space, thus is closed. Thus $(Y \cap C) \cup f(C \cap A) \subset Y$ is a finite union of closed sets, thus is closed. By 8.12, this means $\nu(C)$ is closed, thus ν is a closed map. □

ii) If $z \in X \sqcup_f Y$, show the fiber $\nu^{-1}(z)$ is a non-empty compact subset of $X \sqcup Y$.

Proof. To begin, note that ν is surjective, thus $\nu^{-1}(z)$ is a non-empty set. Note that if $\nu^{-1}(z)$ is a subset of $X \setminus A$ or $Y \setminus f(A)$, then $\nu^{-1}(z) = \{w\}$ for some singleton set. Singleton sets are always compact so we may assume this is not the case and $\nu^{-1}(z)$ intersects A or $f(A)$. By our initial lemmas this also guarentees that $\nu^{-1}(z) \cap (X \setminus A) = \emptyset = \nu^{-1}(z) \cap (Y \setminus f(A))$. Also by our initial lemmas, we see $\nu^{-1}(z) \cap Y$ is a singleton set, as no two elements of Y share an equivalence class. Thus $\nu^{-1}(z) \cap Y = \{y\}$ for some $y \in Y$. Singleton sets are closed and compact in Y as it is Hausdorff, thus $f^{-1}(Y) \subset A$ is a closed subset in a compact set. Further, we have

$$\nu^{-1}(z) = (\nu^{-1}(z) \cap Y) \cup (\nu^{-1}(z) \cap X) = (\nu^{-1}(z) \cap Y) \cup (\nu^{-1}(z) \cap A)$$

but $\nu^{-1}(z) \cap A$ is all those a such that $\nu(a) = z = \nu(y)$, thus it is all those a such that $f(a) = y$, which in turn is just $f^{-1}(y)$. Thus

$$\nu^{-1}(z) = \{y\} \cup f^{-1}(y)$$

which is the finite union of compact sets and thus is compact. $\square$