Prop 2.29, Thm 2B.5 in Hatcher

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Proposition 2.29: *$\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even,* [p. 135, Hatcher].

An **action of a group** G on a space X is a homomorphism from G to the group Homeo(X) of homeomorphisms $X \to X$, and the action is **free** if the homeomorphism corresponding to each nontrivial element of G has no fixed points. In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$ on S^n.

For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ is a homeomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$, where α is the generator of $\tilde{H}_n(S^n)$, for some integer d depending only on f. The integer d is called the **degree** of f with notation $\deg f$.

Some properties of the degree used in the proof:

(a) $\deg \mathbb{1} = 1$ since $\mathbb{1}_* = \mathbb{1}$, where $\mathbb{1}$ is the identity on S^n.

(c) If $f \simeq g$, then $\deg f = \deg g$ since $f_* = g_*$. The converse statement, that $f \simeq g$ if $\deg f = \deg g$, is a fundamental theorem from Hopf (1925) proven in Corollary 4.25 (Hatcher).

(d) $\deg fg = \deg f \deg g$ since $f_* g_* = (fg)_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $fg = \mathbb{1}$ implies $\deg f \deg g = \deg \mathbb{1} = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n fixing the points in a subsphere S^{n-1} and interchanging the two complimentary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

(f) The antipodal map $-\mathbb{1} : S^n \to S^n$, $x \mapsto x$ has degree $(-1)^{n+1}$ since it is a composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.

(g) If $f : S^n \to S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$ since for $f(x) \neq x$, the line segment from $f(x)$ to $-x$ defined by $t \mapsto (1-t)f(x) - tx$ for $0 \leq t \leq 1$ does not pass through the origin, and so, if f has no fixed points, the formula

$$f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

defines a homotopy $f_t : S^n \times I \to S^n$ from f to the antipodal map.

Proof of Proposition 2.29: Since homeomorphisms have degree ± 1 by (d), an action of a group G on S^n determines a degree function $d : G \to \{\pm 1\}$. This is a homomorphism since $\deg(fg) = \deg f \deg g$.

If the action is free, each nontrivial element of G is sent to some homeomorphism $f : S^n \to S^n$ with no fixed points and so by (g), $\deg f = (-1)^{n+1}$. Thus, when n is even, $\ker d = \{1_G\}$, and so, $G = \mathbb{Z}_2$. □

Theorem 2B.3: *If a subspace X of $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{R}^n$, then X itself is open in $\mathbb{R}^n$*, [p. 172, Hatcher].

Corollary 2B.4: *If M is a compact n-manifold and N is a connected n-manifold, then an embedding $M \hookrightarrow N$ must be surjective,* [p. 172, Hatcher].

Theorem 2B.5: *$\mathbb{R}$ and $\mathbb{C}$ are the only finite dimensional division algebras over $\mathbb{R}$ which are commutative and have an identity,* [p. 173 Hatcher].

An **algebra** structure on $\mathbb{R}^n$ is simply a bilinear multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(a, b) \mapsto ab$. Thus the product satisfies left and right distributivity

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc,$$

and scalar associativity

$$\alpha(ab) = (\alpha a)b = a(\alpha b), \ \forall \alpha \in \mathbb{R}.$$

Commutativity, full associativity, and an identity element are not assumed.

An algebra is a **division algebra** if the equations $ax = b$ and $xa = b$ are always solvable whenever $a \neq 0$. These are linear maps $\mathbb{R}^n \to \mathbb{R}^n$ so surjectivity is equivalent to having trivial kernel, meaning there are no (nonzero) zero-divisors.

The four classical examples are $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. Frobenius (1877) proved that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only finite dimensional associative division algebras over $\mathbb{R}$, and Hurwitz (1898) proved that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only finite dimensional divison algebras over $\mathbb{R}$ with a product satisfying $|ab| = |a||b|$, i.e. the only finite dimensional **normed** division algebras over $\mathbb{R}$.

Proof of Theorem 2B.5: Suppose $\mathbb{R}^n$ has a commutative division algebra structure. Let $f : S^{n-1} \to S^{n-1}$ be defined by $f(x) = \dfrac{x^2}{|x^2|}$. This is well-defined since $x \neq 0$ implies that $x^2 \neq 0$ in a division algebra. This map is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear (hence continuous). Since $f(-x) = f(x)$ for all x, f induces a quotient map $\bar{f} : \mathbb{RP}^{n-1} \to S^{n-1}$.

For $x,\, y \in S^{n-1}$, $f(x) = f(y)$ implies $x^2 = \alpha^2 y^2$ for $\alpha = \left(\dfrac{|x^2|}{|y^2|}\right)^{1/2} > 0$.

Thus, $x^2 - \alpha^2 y^2 = 0$ and $(x + \alpha y)(x - \alpha y) = 0$ using commutativity and the fact that α is a real scalar (scalar associativity).

Since there are no (nonzero) zero-divisors, $x = \pm\alpha y$, and since x, y are unit vectors and α is real, $x = \pm y$. So, x and y determine the same point of $\mathbb{RP}^{n-1}$ and so $\bar{f}$ is injective.

Since $\bar{f}$ is continuous injective and $\mathbb{RP}^{n-1}$, S^{n-1} are compact Hausdorff spaces, $\bar{f}$ is a closed continuous injective map and hence is a homeomorphism onto its image.

$\bar{f}$ must also be surjective if $n > 1$ by **Corollary 2B.4**. Thus, we have a homeomorphism $\mathbb{RP}^{n-1} \cong S^{n-1}$, and so $n = 2$ since otherwise $\mathbb{RP}^{n-1}$ and S^{n-1} have different fundamental groups and different homology groups when $n > 2$.

If $n = 1$, then $\mathbb{RP}^0 \cong S^0$ since $\mathbb{RP}^0$ is th set of lines that pass through the origin in $\mathbb{R}^{0+1}$, which is only $\mathbb{R}$ itself, and $S^0 = \{1\}$.

Thus, the only finite dimensional commutative division algebras over $\mathbb{R}$ with identity are either dimension 1 or 2.

It remains to show that a 2-dimensional commutative division algebra A with identity is isomorphic to $\mathbb{C}$, which only requires some elementary algebra. Let $j \in A$ such that $j \neq \alpha 1$ for any $\alpha \in \mathbb{R}$, 1 being the identity in A.

Then, $j^2 = a + bj$ for some $a,\, b \in \mathbb{R}$ since A is 2-dimensional over $\mathbb{R}$ and $1,\, j$ are linearly independent. Thus, $\left(j - \frac{b}{2}\right)^2 = a + \frac{b^2}{4}$ so by rechoosing j, we can assume $j^2 = a \in \mathbb{R}$. If $a \geq 0$, say $a = c^2$ for $c \in \mathbb{R}$. So,

$$j^2 = c^2 \Rightarrow (j + c)(j - c) = 0 \Rightarrow j = \pm c,$$

but this contradicts our choice of j such that $j \neq \alpha 1$ for all $\alpha \in \mathbb{R}$. So, $j^2 = -c^2$, and by rescaling j, we may assume $j^2 = -1$. So, $A \cong \mathbb{C}$. $\square$