Math 663

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Let $n \in \mathbb{N}$. Let $y \in \mathbb{S}^n$, and f a map $f: \mathbb{S}^n \to \mathbb{S}^n$ such that $f^{-1}(\{y\})$ is finite.

Let $m \in \mathbb{N}$ such that $f^{-1}(\{y\}) = \{x_1, x_2, \dots, x_i, \dots, x_m\}$. Let V and open neighborhood of y for all i U_i are pairwise disjoint open neighborhoods of x_i, respectively, and $f(U_i) \subseteq V$.

Then, for all i $f|_{U_i - \{x_i\}}: U_i - \{x_i\} \to V - \{y\}$ is a map that induces a homology homomorphism, ${f_i}_*: H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\})$.

Detail 1.

In the long exact sequence $\cdots \to H_n(\mathbb{S}^n - \{x\}) \overset{i_*}{\to} H_n(\mathbb{S}^n) \overset{j_*}{\to} H_n(\mathbb{S}^n, \mathbb{S}^n - \{x\}) \overset{\partial}{\to} H_{n-1}(\mathbb{S}^n - \{x\}) \to \cdots$, $\mathbb{S}^n - \{x\}$ is contractible, so for all $n > 0$, $H_n(\mathbb{S}^n - \{x\}) \cong 0$. Hence, $0 \to H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n, \mathbb{S}^n - \{x\}) \to 0$ is a short exact sequence. This implies that $H_n(\mathbb{S}^n, \mathbb{S}^n - \{x\}) \cong H_n(\mathbb{S}^n) \cong \mathbb{Z}$.

For all $x \in \mathbb{S}^n$ and open neighborhood U of x. $\{x\}$ is closed, so $\mathbb{S}^n - \{x\}$ is open, U is open, so $\mathbb{S}^n - U$ is closed. $\{x\} \subseteq U$ implies that $\mathbb{S}^n - U \subseteq \mathbb{S}^n - \{x\}$; therefore, by Excision Theorem, $H_n(U, U - \{x\}) = H_n\big(\mathbb{S}^n - (\mathbb{S}^n - U), (\mathbb{S}^n - \{x\}) - (\mathbb{S}^n - U)\big) \cong H_n(\mathbb{S}^n, \mathbb{S}^n - \{x\}) \cong \mathbb{Z}$.

It follows that f_{i_*} is a group homomorphism between infinite cyclic groups.

The local degree of f at x_i is given by $\deg f|_{x_i} = f_{i_*}(1)$

Proposition 2.30.

Let $\deg f = \sum_i \deg f|_{x_i}$.

Proof.

The set $\{x_1, \dots, x_i, \dots, x_m\} = f^{-1}(\{y\})$ is closed, so $\mathbb{S}^n - f^{-1}(\{y\})$ is open. $\bigcup_{i=1}^m U_i$ is also open, so $\mathbb{S}^n - \bigcup_{i=1}^m U_i$ is closed and $\mathbb{S}^n - \bigcup_{i=1}^m U_i \subseteq \mathbb{S}^n - f^{-1}(\{y\})$; therefore, by

Excision Theorem, $H_n(\coprod_{i=1}^m U_i, \coprod_{i=1}^m (U_i - \{x_i\})) = H_n(\coprod_{i=1}^m U_i, \coprod_{i=1}^m U_i - \bigcup_{i=1}^m \{x_i\}) = H_n\left(\mathbb{S}^n - (\mathbb{S}^n - \bigcup_{i=1}^m U_i), \left(\mathbb{S}^n - f^{-1}(\{y\})\right) - (\mathbb{S}^n - \bigcup_{i=1}^m U_i)\right) \cong H_n\left(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(\{y\})\right)$.

Each U_i is a disconnection of $\coprod_{i=1}^m U_i$, so in particular it is a path-disconnection. Then, by proposition 2.6, $H_n(\coprod_{i=1}^m U_i, \coprod_{i=1}^m (U_i - \{x_i\})) \cong \bigoplus_{i=1}^m H_n(U_i, U_i - \{x_i\}) \cong \bigoplus_{i=1}^m \mathbb{Z}_i$.

$f|_{\mathbb{S}^n - f^{-1}(\{y\})}$ is a map such that $f|_{\mathbb{S}^n - f^{-1}(\{y\})}: \mathbb{S}^n - f^{-1}(\{y\}) \to \mathbb{S}^n - \{y\}$, so there is an induced group homomorphism $g_*: H_n\left(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(\{y\})\right) \to H_n(\mathbb{S}^n, \mathbb{S}^n - \{y\})$.

Let the following diagrams.

$$\begin{array}{ccccc} & & H_n(U_i,U_i-\{x_i\}) & \overset{f_{i_*}}{\to} & H_n(V,V-\{y\}) \\ & \swarrow_{\cong} & & & \updownarrow ET \\ H_n(\mathbb{S}^n,\mathbb{S}^n-\{x_i\}) & & & & H_n(\mathbb{S}^n,\mathbb{S}^n-\{y\}) \\ & \nwarrow_{\cong} & & & \updownarrow LES \\ & & H_n(\mathbb{S}^n) & \overset{f_*}{\to} & H_n(\mathbb{S}^n) \end{array}$$

Pic 1

$$\begin{array}{ccccc} & & H_n(U_i,U_i-\{x_i\}) & \overset{f_{i_*}}{\to} & H_n(V,V-\{y\}) \\ & \swarrow_{\cong} & \downarrow \iota_i & & \updownarrow ET \\ H_n(\mathbb{S}^n,\mathbb{S}^n-\{x_i\}) & \overset{p_i}{\leftarrow} & \bigoplus_{i=1}^{m} H_n(U_i,U_i-\{x_i\}) & & H_n(\mathbb{S}^n,\mathbb{S}^n-\{y\}) \\ & \nwarrow_{\cong} & \uparrow j_* & & \updownarrow LES \\ & & H_n(\mathbb{S}^n) & \overset{f_*}{\to} & H_n(\mathbb{S}^n) \end{array}$$

Pic 2

$$\begin{array}{ccccc} & & H_n(U_i,U_i-\{x_i\}) & \overset{f_{i_*}}{\to} & H_n(V,V-\{y\}) \\ & \swarrow_{\cong} & \downarrow \iota_i & & \updownarrow ET \\ H_n(\mathbb{S}^n,\mathbb{S}^n-\{x_i\}) & \overset{p_i}{\leftarrow} & H_n\left(\mathbb{S}^n,\mathbb{S}^n-f^{-1}(\{y\})\right) & \overset{g_*}{\to} & H_n(\mathbb{S}^n,\mathbb{S}^n-\{y\}) \\ & \nwarrow_{\cong} & \uparrow j_* & & \updownarrow LES \\ & & H_n(\mathbb{S}^n) & \overset{f_*}{\to} & H_n(\mathbb{S}^n) \end{array}$$

Pic 3

$\bigoplus_{i=1}^m H_n(U_i, U_i - \{x_i\}) \cong H_n\left(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(\{y\})\right)$ yields the next diagram.

$$\begin{array}{ccccc} & & H_n(U_i, U_i - \{x_i\}) & \overset{f_{i_*}}{\rightarrow} & H_n(V, V - \{y\}) \\ & \swarrow_{\cong} & \downarrow \iota_i & & \updownarrow ET \\ H_n(\mathbb{S}^n, \mathbb{S}^n - \{x_i\}) & \overset{p_i}{\leftarrow} & \bigoplus_{i=1}^{m} H_n(U_i, U_i - \{x_i\}) & \overset{g_*}{\rightarrow} & H_n(\mathbb{S}^n, \mathbb{S}^n - \{y\}) \\ & \nwarrow_{\cong} & \uparrow j_* & & \updownarrow LES \\ & & H_n(\mathbb{S}^n) & \overset{f_*}{\rightarrow} & H_n(\mathbb{S}^n) \end{array}$$

Pic 4

In the diagram, the maps ι_i and p_i are induced by the inclusion map. The map j_* is induced by the quotient map. Hence, the squares and triangles in the diagrams commute.

Passing the former diagram to its isomorphic images yields.

$$\begin{array}{ccccc} & & \mathbb{Z}_i & \overset{f_{i_*}}{\rightarrow} & \mathbb{Z} \\ & \swarrow_{\cong} & \downarrow \iota_i & & \updownarrow ET \\ \mathbb{Z}_i & \overset{p_i}{\leftarrow} & \bigoplus_{i=1}^{m} \mathbb{Z}_i & \overset{g_*}{\rightarrow} & \mathbb{Z} \\ & \nwarrow_{\cong} & \uparrow j_* & & \updownarrow LES \\ & & \mathbb{Z} & \overset{f_*}{\rightarrow} & \mathbb{Z} \end{array}$$

Pic 5

ι_i is the inclusion homomorphism, p_i is the projection homomorphism, and $j_*(1) = \prod_{i=1}^{m}\{1\}_i$.

Due to the commutativity of the upper square, $g_*\big(\iota_i(1)\big) = \pm f_{i_*}(1) = \pm \deg f|_{x_i}$. Due to the commutativity of the lower square, $g_*\big(j_*(1)\big) = \pm f_*(1) = \pm \deg f$, where $g_*\big(j_*(1)\big) = g_*(\prod_{i=1}^{m}\{1\}_i) = g_*\big(\oplus_{i=1}^{m} \iota_i(1)\big) = \oplus_{i=1}^{m} g_*\big(\iota_i(1)\big) = \sum_{i=1}^{m} \pm \deg f|_{x_i}$.

References.

Hatcher, A. (2001). Algebraic Topology.