MORSE THEORY NOTES

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The main resource for this set of notes is Hirsch's *Differential Topology*, with only minor modifications made to several of the main proofs. Other resources used include Milnor's *Morse Theory* and Guillemin and Pollack's *Differential Topology*.

1. Morse Functions

Recall that a point $x \in M$ is a *regular point* of a smooth map $f : M \to N$ if the derivative of f at x, $df_x : T_x M \to T_{f(x)} N$ is surjective, i.e. f is a submersion at x. Otherwise, x is said to be a *critical point* of f. A point $y \in N$ is a *regular value* of f if every $x \in f^{-1}(y)$ is a regular point. Otherwise, y is a *critical value*.

Note 1.1. For a real-valued smooth function $f : M \to \mathbb{R}$, p is a critical point of f if and only if $df_p : T_x M \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is the zero map. In terms of local coordinates $(x_1, \ldots, x_n)$ for a neighborhood of p, this means that the partial derivatives $\frac{\partial f}{\partial x_i}$ all vanish at p.

Definition 1.2. If p is a critical point of $f : M \to \mathbb{R}$ and $(x_1, \ldots, x_n)$ is a coordinate system for a neighborhood of p, the *Hessian matrix* of f at p with respect to the coordinates $(x_1, \ldots, x_n)$ is the $n \times n$ matrix of second partial derivatives

$$H_f(p) = \Big(\frac{\partial f}{\partial x_i \partial x_j}(p)\Big).$$

Lemma 1.3. *Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two coordinates systems for a critical point p, and let $H_f(p)$ and $H_f'(p)$ be the Hessians of f with respect to these coordinate systems respectively. If*

$$J(p) = \Big(\frac{\partial x_i}{\partial y_j}(p)\Big)$$

is the Jacobian matrix of the coordinate transformation from $(y_1, \ldots, y_n)$ to $(x_1, \ldots, x_n)$ then

$$H_f'(p) = J(p)^T H_f(p) J(p).$$

Proof. By the chain rule, we have that

$$\frac{\partial f}{\partial y_h} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_h}.$$

Hence

$$\frac{\partial^2 f}{\partial y_h \partial y_k}(p) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\frac{\partial x_i}{\partial y_h}(p)\frac{\partial x_j}{\partial y_k}(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\frac{\partial x_i}{\partial y_h \partial y_k}(p).$$

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However, by the assumption that p is a critical point of f, all of the terms in the second sum vanish, i.e.

$$\frac{\partial^2 f}{\partial y_h \partial y_k}(p) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \frac{\partial x_i}{\partial y_h}(p) \frac{\partial x_j}{\partial y_k}(p).$$

It follows that

$$H'_f(p) = J(p)^T H_f(p) J(p).$$

□

Definition 1.4. A critical point p of $f : M \to \mathbb{R}$ is *non-degenerate* if the Hessian matrix of f at p is non-singular with respect to some coordinate system at p.

Remark 1.5. The previous definition is well-defined since if $H_f(p)$ is non-singular for some coordinate system, Lemma 1.3 implies that $H_f(p)$ is non-singular with respect to any coordinate system. This follows by the fact that coordinate transformations are diffeomorphisms, hence $\det J(p) \neq 0$ and so

$$\det H'_f(p) = \det\Big(J(p)^T H_f(p) J(p)\Big) = \det J(p) \det H_f(p) \det J(p)$$

is nonzero if and only if $\det H_f(p)$ is nonzero.

Example 1.6.

Definition 1.7. We say that $f : M \to \mathbb{R}$ is a *Morse function* if every critical point of f is non-degenerate.

Before we continue to study Morse functions, one should check that they in fact always exist. This is true, and even better, Morse functions are *generic*. For any finite dimensional manifold M, Whitney's embedding theorem ensures that M can be embedded in $\mathbb{R}^N$ for N sufficiently large. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^N$, and for any smooth function $f : M \to \mathbb{R}$ and $a \in \mathbb{R}^N$, define $f_a : M \to \mathbb{R}$ by $f_a(x) = f(x) + \langle x, a \rangle$.

Proposition 1.8. *For any smooth function $f : M \to \mathbb{R}$, f_a is a Morse function for almost every $a \in \mathbb{R}^N$.*

The proof of the above proposition involves using Sard's theorem to show it holds when M is an open subset of $\mathbb{R}^n$ and then extending the result to the general case by taking a countable open cover of M by charts. Details can be found in *Differential Topology* by Guillemin and Pollack on page 43.

Definition 1.9. The *index* of $f : M \to \mathbb{R}$ at a non-degenerate critical point p is the number of negative eigenvalues of $H_f(p)$ and is denoted by $\mathrm{Ind}_f(p)$.

Being defined in terms of the eigenvalues of the Hessian matrix, we should check that the index of f at p does not depend on the choice of coordinates. This follows from Lemma 1.3 and Sylvester's Law. Recall that two $n \times n$ matrices A and B are *similar* if there exists an invertible matrix P such that $A = P^T BP$. Sylvester's Law states that similar matrices have the same number of negative and positive eigenvalues. Lemma 1.3 shows that if $H'_f(p)$ and $H_f(p)$ are Hessian matrices of f at p with respect to different coordinate systems, then $H'_f(p)$ and $H_f(p)$ are similar. Thus the index of f at p is independent of the choice of coordinates.

Remark 1.10. If we we interpret a Morse function $f : M \to \mathbb{R}$ as representing a "height" function on M, then the index of f at a critical point p can be understood as the number of independent directions for which M curves downward away from p. Alternatively, this is the number of independent directions for which the sublevel sets $M^a = \{x \in M : f(x) \leq a\}$ approach the point p as a increases to $f(p)$.

Example 1.11.

Lemma 1.12 (Morse's Lemma)**.** *Let $p \in M$ be a non-degenerate critical point of index k of a smooth function $f : M \to \mathbb{R}$. Then there exists a chart (ϕ, U) at p such that*

$$f\phi^{-1}(y_1, y_2, \ldots, y_n) = f(p) - \sum_{i=1}^{k} y_i^2 + \sum_{i=k+1}^{n} y_i^2.$$

Before we can prove Morse's lemma, we first need the following linear algebra lemma.

Lemma 1.13. *Let $A = \operatorname{diag}\{a_1, \ldots, a_n\}$ be a diagonal $n \times n$ matrix with diagonal entries ± 1. Then there exists a neighborhood N of A in the vector space of symmetric $n \times n$ matrices, and a smooth map $P : N \to GL(n, \mathbb{R})$ such that $P(A) = I$ (the identity matrix) and if $P(B) = Q$, then $Q^T BQ = A$.*

Proof. Let $B = (b_{ij})$ be a symmetric matrix near enough to A so that b_{11} is nonzero and has the same sign as a_1. Consider the linear change of coordinates defined by $x = Ty$ for which

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{|b_{11}|}}\Big(y_1 - \frac{b_{12}}{b_{11}} y_2 - \cdots - \frac{b_{1n}}{b_{11}} y_n\Big) \\ x_k &= y_k \qquad \text{for} \qquad k = 2, \ldots, n. \end{aligned}$$

A calculation shows that

$$T^T BT = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix}$$

If B is near enough to A then the symmetric $(n-1)\times(n-1)$ matrix B_1 will be close as desired to the diagonal matrix $A_1 = \operatorname{diag}\{a_2, \ldots, a_n\}$; in particular it will be invertible. Note that T and B_1 are smooth functions of B in a neighborhood of A. By induction on n, we assume there exists a matrix $Q_1 = P_1(B_1) \in GL(n-1)$ depending analytically on B_1, such that $Q_1^T B_1 Q_1 = A_1$. Define $P(B) = Q$ by $Q = TS$ where

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Q_1 & \\ 0 & & & \end{pmatrix}.$$

Then $Q^T BQ = S^T(T^T BT)S = A$. □

Proof of Morse's Lemma. By restricting to some chart at p, we may assume without loss of generality that M is a convex open subset of $\mathbb{R}^n$ and that $p = 0$. By replacing f with $f - f(0)$ if needed, we may assume that $f(0) = 0$. Furthermore, by a linear coordinate change, we may suppose that the matrix

$$A = \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \Big)$$

is diagonal, with -1 as the first k diagonal entries, and $+1$ for all others.

We first show that there exists a smooth map $x \mapsto B_x$ from M to the space of symmetric $n \times n$ matrices such that if we write $B_x = (b_{ij}(x))$, then

$$f(x) = \sum_{i,j=1}^{n} b_{ij}(x) x_i x_j$$

and for which $B_0 = A$. Note that by the fundamental theorem of calculus and the fact that $f(0) = 0$, we have

$$\begin{aligned} f(x) &= f(x) - f(0) \\ &= \int_0^1 \frac{df(tx)}{dt}\, dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx) x_i \, dt \\ &= \sum_{i=1}^{n} \Big(\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt \Big) x_i \end{aligned}$$

Since $p = 0$ is a critical point of f, $\frac{\partial f}{\partial x_i}(0) = 0$ for all i. Hence the same argument shows that for all i and all $t \in [0, 1]$,

$$\frac{\partial f}{\partial x_i}(tx) = \sum_{j=1}^{n} \Big(\int_0^1 \frac{\partial f}{\partial x_i \partial x_j}(stx)\, ds \Big) x_j.$$

Thus
$$f(x) = \sum_{i,j=1}^{n} \Big(\int_0^1 \int_0^1 \frac{\partial f}{\partial x_i \partial x_j}(stx)\, ds\, dt \Big) x_i x_j,$$
which we may write as
$$f(x) = \sum_{i,j=1}^{n} b_{ij}(x) x_i x_j.$$
By construction, $B_0 = A$ and the $b_{ij}(x)$ are smooth functions of x, thus so too is $x \mapsto B_x$.

Let $P : N \to GL(n, \mathbb{R})$ be the matrix valued function from the previous lemma and set $P(B_x) = Q_x$. Define a smooth map $\phi : U \to \mathbb{R}^n$ by $\phi(x) = Q_x^{-1} x$ on a neighborhood U of 0. A calculation shows that $d\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, hence by the inverse function theorem ϕ is a local diffeomorphism at 0. Indeed, writing $Q_x^{-1} = (q_{ij}(x))$ we have that
$$\phi(x) = \Big(\sum_{k=1}^{n} q_{1k}(x) x_k, \dots, \sum_{k=1}^{n} q_{nk}(x) x_k \Big).$$
Then
$$\begin{aligned} \frac{\partial \phi_i}{\partial x_j}(x) &= \frac{\partial}{\partial x_j} \Big(\sum_{k=1}^{n} q_{ik}(x) x_k \Big) \\ &= \sum_{k=1}^{n} \Big(\frac{\partial q_{ik}}{\partial x_j}(x) x_k + q_{ik}(x) \delta_{kj} \Big) \end{aligned}$$
where δ is the Kronecker delta. Evaluating at $x = 0$ we see that
$$\frac{\partial \phi_i}{\partial x_j}(0) = q_{ij}(0).$$
Hence the Jacobian matrix of ϕ at 0 is simply $Q_0^{-1} = P(B_0)^{-1} = P(A)^{-1} = I$ and so $d\phi_0$ is the identity. Thus we may take U to be small enough so that ϕ is a diffeomorphism onto its image, in which case (ϕ, U) is a smooth chart at 0. Finally, set $y = \phi(x)$. Then
$$\begin{aligned} f(x) &= x^T B_x x \\ &= (Q_x y)^T B_x (Q_x y) \\ &= y^T (Q_x^T B_x Q_x) y \\ &= y^T A y \\ &= \sum_{i=1}^{n} a_{ii} y_i^2. \end{aligned}$$
□

Corollary 1.14. *Non-degenerate critical points of a smooth function $f : M \to \mathbb{R}$ are isolated from other critical points of f.*

Proof. By Morse's Lemma, there exists a chart (ϕ, U) at a critical point p such that
$$f\phi^{-1}(x_1, \dots, x_n) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

Then the i-th partial derivative of $f\phi^{-1}$ is simply $\pm 2x_i$. Hence $0 \in \phi(U)$ is the only point for which the partial derivatives of $f\phi^{-1}$ all vanish. Thus the only critical point of f in U is $p = \phi^{-1}(0)$. □

Corollary 1.15. *A Morse function on a compact manifold admits only finitely many critical points.*

Proof. All critical points of f are isolated by the previous corollary. If the set of critical points of f is closed in M, then there exists an open cover of M so that each element of the open cover contains at most one critical point of f. By compactness, there exists a finite subcover, which implies the number of critical points is finite.

To see that the set of critical points of f is closed, note that the map $p \mapsto df_p$ from M to the space of linear maps $\mathbb{R}^n \to \mathbb{R}$ is smooth and the set of critical points of f is precisely the preimage of the zero map $\mathbb{R}^n \to \mathbb{R}$ under $p \mapsto df_p$. □

2. Regular Interval Theorem

With the local characterization of f at non-degenerate critical points given by Morse's Lemma, we now turn to analyzing how the topology of the sublevel sets M^a changes as a increases. This is accomplished in two main steps. First we show that if an interval $[a, b]$ contains no critical points, then M^a is a deformation retract of M^b. Hence the two sublevel sets are homotopy equivalent. Second, we show that if $[a, b]$ contains a single critical value $c \in (a, b)$ and $f^{-1}(c)$ consists of a single critical point p of index k, then there exists a deformaton retraction of M^b onto $M^a \cup e^k$, where e^k is a k-cell. The proofs of both steps rely on *integral curves* of the *gradient vector field* of f.

Definition 2.1. Let $X : M \to TM$ be a smooth vector field on M. An *integral curve* (or *solution curve*) of X is a differentiable map $\eta : J \to M$ where $J \subset \mathbb{R}$ is an interval and $\eta'(t) = X(\eta(t))$. Here $\eta'(t)$ denotes the image of the tangent vector $1 \in \mathbb{R} \cong T_tJ$ under $d\eta_t : T_tJ \to T_{\eta(t)}M$.

Remark 2.2. The existence and uniqueness of integral curves is locally guarenteed by the Picard-Lindelöf theorem.

Definition 2.3. Given a vector field X on M, for each $x \in M$, a *trajectory* (or *flowline*) of X is a solution curve $\eta^x : J(x) \to M$ where $\eta^x(0) = x$ and $J(x)$ is the maximal interval about 0 for which $(\eta^x)'(t) = X(\eta^x(t))$.

Assume M has a smooth Riemannian metric, that is, the tangent space T_pM at each point $p \in M$ is equipped with an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ such that the map $(p, X, Y) \mapsto g_p(X, Y)$ defined on $\{(p, X, Y) \in M \times TM \times TM : X, Y \in T_pM\}$ is smooth. Unless it causes confusion, we denote the inner product on any T_pM by $\langle X, Y\rangle$. The corresponding norm is $|X| = \langle X, X\rangle^{1/2}$.

For every linear map $\lambda : T_pM \to \mathbb{R}$ there exists a unique tangent vector $X_\lambda \in T_pM$, called the *dual* to λ, satisfying $\lambda(Y) = \langle X_\lambda, Y\rangle$.

Definition 2.4. If $f : M \to \mathbb{R}$ is smooth, define $\operatorname{grad} f(p) \in T_pM$ to be the dual of $df_p : T_pM \to \mathbb{R}$. The vector field $\operatorname{grad} f : M \to TM$ is then naturally defined by $p \mapsto \operatorname{grad} f(p)$.

Example 2.5. If M is an open subset of $\mathbb{R}^n$ and the Riemannian metric is given by the standard inner product on $\mathbb{R}^n$, then

$$\operatorname{grad} f(x) = \Big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\Big).$$

Definition 2.6. A *gradient line* $\eta : J \to M$ of $f : M \to \mathbb{R}$ is a solution curve of the gradient vector field $\operatorname{grad} f$.

The following elementary observations about $\operatorname{grad} f$ follow immediately from the definitions.

Proposition 2.7. *A point $p \in M$ is a critical point of f if and only if* $\operatorname{grad} f(p) = 0$.

Proof. If p is a critical point of $f : M \to \mathbb{R}$, then df_p is the zero map. Thus $df_p(Y) = \langle \operatorname{grad} f(p), Y\rangle = 0$ for all $Y \in T_pM$. In particular

$$\langle \operatorname{grad} f(p), \operatorname{grad} f(p)\rangle = 0,$$

implying that $\operatorname{grad} f(p) = 0$. Conversely, if $\operatorname{grad} f(p) = 0$, then $df_p(Y) = \langle 0, Y\rangle = 0$ for all $Y \in T_pM$, hence p is a critical point of f. □

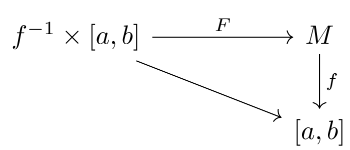
Proposition 2.8. *Let $\eta : J \to M$ be a gradient line of f. Then $f\eta : J \to \mathbb{R}$ is nondecreasing. Moreover, if $\eta(J)$ contains no critical points of f, then $f\eta$ is strictly increasing and is thus transverse to the level sets $f^{-1}(f(\eta(t)))$.*

Proof. To see that $f\eta$ is nondecreasing, note that

$$\begin{aligned}\frac{d}{dt} f(\eta(t)) &= df_{\eta(t)}(\eta'(t)) \\ &= \langle \operatorname{grad} f(\eta(t), \operatorname{grad} f(\eta(t))\rangle \\ &= |\operatorname{grad} f(\eta(t))|^2 \geq 0\end{aligned}$$

If $\eta(J)$ contains no critical points of f, then $\operatorname{grad} f(\eta(t)) \neq 0$ for all $t \in J$, and the above work shows that $f\eta$ is strictly increasing. □

Theorem 2.9 (Regular Interval Theorem)**.** *Let $f : M \to [a,b]$ be a smooth map on a compact manifold with boundary. Suppose that f has no critical points and $f(\partial M) = \{a, b\}$. Then there is a diffeomorphism $F : f^{-1}(a) \times [a,b] \to M$ so that the diagram*



commutes. In particular, all the level surfaces of f are diffeomorphic.

Proof. Give M a Riemannian metric. Define the vector field $X : M \to TM$ by

$$X(x) = \frac{\operatorname{grad} f(x)}{|\operatorname{grad} f(x)|^2}.$$

Note that the solution curves of X are simply the solution curves of $\operatorname{grad} f$ but with a different parameterization. If $\eta : [t_0, t_1] \to M$ is a solution curve of X, then the derivative of $f \circ \eta$ is

$$\begin{aligned}\frac{d}{dt} f \circ \eta &= \langle \operatorname{grad} f(\eta(t)), X(\eta(t)) \rangle \\ &= \frac{1}{|\operatorname{grad} f(\eta(t))|^2} \langle \operatorname{grad} f(\eta(t)), \operatorname{grad} f(\eta(t)) \rangle \\ &= 1.\end{aligned}$$

Hence

$$f(\eta(t_1)) - f(\eta(t_0)) = t_1 - t_0. \tag{1}$$

Let $x \in f^{-1}(s)$. Since M is compact, the set $J(x)$ is closed. Hence by (1),

$$J(x) = [a - s, b - s]. \tag{2}$$

Since a is a regular value and $f(\partial M) = \{a, b\}$, $f^{-1}(a)$ is a union of boundary components of M. Define a map $F : f^{-1}(a) \times [a, b] \to M$ by

$$F(x, t) = \eta^x(t - a).$$

We now show that F is a diffeomorphism. If $F(x_1, t_1) = F(x_2, t_2)$, uniqueness of solution curves implies that $x_1 = x_2$. Since f increases along gradient lines, it also increases along the trajectories of X, showing that $t_1 < t_2$ implies that $F(x, t_1) < F(x, t_2)$. Thus F is injective. Because gradient lines are tranverse to level sets, F is also an immersion. Hence F is an embedding. Lastly, (2) implies that F is surjective. □

Example 2.10.

Example 2.11. To emphasize the role of compactness in the proof of the Regular Interval Theorem, consider the following example of a manifold M which is not compact but otherwise satisfies the conditions of the theorem. The function $f : M \to \mathbb{R}$ is the height function on M as depicted.

3. Passing Critical Levels

Definition 3.1. A Morse function $f : M \to [a, b]$ is *admissible* if $\partial M = f^{-1}(a) \cup f^{-1}(b)$ and both a and b are regular values of f.

Example 3.2.

Recall that a k-cell e^k in M is the image of an embedding $D^k \to M$

Theorem 3.3. *Let M be compact and $f : M \to [a, b]$ an admissible Morse function. Suppose f has a unique critical point p, of index k. Then there exists a k-cell $e^k \subset M - f^{-1}(b)$ such that $e^k \cap f^{-1}(a) = \partial e^k$, and there exists a deformation retraction of M onto $f^{-1}(a) \cup e^k$.*

Proof. Let $f(p) = c$, $a < c < b$. To prove the theorem it suffices to prove it for the restriction of f to $f^{-1}[a', b']$ for any a', b' satisfying $a < a' < c < b' < b$ by applying the Regular Interval Theorem to $f^{-1}[a, a']$ and $f^{-1}[b, b']$. Moreover, we can assume that $c = 0$ by replacing f with $f - c$ otherwise.

Let (ϕ, U) be a chart at p as in Morse's lemma. Write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then ϕ maps U diffeomorphically onto an open set $V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ and

$$f \circ \phi^{-1}(x, y) = -|x|^2 + |y|^2,$$

where $(x, y) \in V$. Note that $\phi(p) = (0, 0)$. For ease of notation, set $g = f \circ \phi^{-1}$.

Choose $0 < \delta < 1$ so that V contains the set $\Gamma = D^k(\delta) \times D^{n-k}(\delta)$ where $D^i(\delta) \subset \mathbb{R}^i$ denotes the closed ball centered at 0 of radius δ. Give M a Riemannian metric which agrees in $\phi^{-1}(\Gamma)$ with the metric induced by ϕ from the standard inner product on $\mathbb{R}^n$. That is, if $u \in \phi^{-1}(\Gamma)$ and $X, Y \in T_u M$, then define an inner product on $T_u M$ by

$$\langle X, Y \rangle = \langle d\phi_u(X), d\phi_u(Y) \rangle$$

where the inner product on the right is the standard inner product on $\mathbb{R}^n$. If $\phi(u) = v \in \Gamma$, then

$$d\phi_u(\operatorname{grad} f(u)) = \operatorname{grad} g(v).$$

Choose $\varepsilon > 0$ so that $\sqrt{4\varepsilon} < \delta$. Set

$$\begin{aligned} B^k &= D^k(\sqrt{\varepsilon}) \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \\ &= \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x|^2 \leq \varepsilon\}. \end{aligned}$$

and let $e^k = \phi^{-1}(B^k)$. A deformation of $f^{-1}[-\varepsilon, \varepsilon]$ to $f^{-1}(-\varepsilon) \cup e^k$ is made by patching together two deformations. First consider the set

$$\Gamma_1 = D^k(\sqrt{\varepsilon}) \times D^{n-k}(\sqrt{2\varepsilon}).$$

The set Γ_1 was chosen so that the level set $g^{-1}(\epsilon)$ intersects $\partial\Gamma_1$ in a particular manner, as shown in the below figure. Indeed, if $(x, y) \in \Gamma_1$ satisfies $|x| = \sqrt{\varepsilon}$ and $|y| = \sqrt{2\varepsilon}$, then

$$g(x, y) = -|x|^2 + |y|^2 = -\varepsilon + 2\varepsilon = \varepsilon.$$

Note that in Γ_1, $g(x, y) = -|x|^2 + |y|^2 \geq \varepsilon + |y|^2 \geq -\varepsilon$. Also, since $|x| \leq \varepsilon$ in Γ_1, we have that $(x, 0) \in B^k$ for all $x \in \Gamma_1$.

In $\Gamma_1 \cap g^{-1}[-\varepsilon, \varepsilon]$, a deformation is obtained by moving a point (x, y) at constant speed along the interval joining (x, y) with $(x, 0) \in B^k$ by $(x, (1-t)y)$. Note that these intervals are the closures of the solution curves of the vector field

$$X(x, y) = (0, -2y).$$

Conjugating this deformation, say H, by ϕ then induces a deformation

$$H'(x, t) = \phi^{-1}(H(\phi(x), t))$$

of $f^{-1}[-\varepsilon, \varepsilon] \cap \phi^{-1}(\Gamma_1)$ onto e^k

Let

$$\Gamma_2 = D^k(\sqrt{2\varepsilon}) \times D^{n-k}(\sqrt{3\varepsilon}).$$

On $f^{-1}[-\varepsilon, \varepsilon] - \phi^{-1}(\Gamma_2)$ our deformation moves each point at constant speed along the flow line of the vector field $-\operatorname{grad} f$ so that it reaches $f^{-1}(-\varepsilon)$ in unit time. (The speed of each point is the length of its path under the deformation.) To see that each flowline of $-\operatorname{grad} f$ starting outside of $\phi^{-1}(\Gamma_2)$ *will* reach $f^{-1}(\varepsilon)$, note that $|\operatorname{grad} f|$ has a positive lower bound in the compact set $f^{-1}[-\varepsilon, \varepsilon] - \operatorname{Int} \phi^{-1}(\Gamma_2)$, and f decreases along the flow lines of $-\operatorname{grad} f$. Hence it suffices to show that any flowline of $-\operatorname{grad} f$ may not enter $\phi^{-1}(\Gamma_2) \cap f^{-1}[-\varepsilon, \varepsilon]$ from outside. This follows from that fact that flowlines of $-\operatorname{grad} f$ in $\phi^{-1}(\Gamma) - \phi^{-1}(\Gamma_2)$ are mapped by ϕ to flowlines of $-\operatorname{grad} g$ in $\Gamma - \Gamma_2$ and $|x|$ increases along any such flowline.

To extend this deformation to points of $\Gamma_2 - \Gamma_1$, it suffices to find a vector field which agrees with X in Γ_1, and with $-\operatorname{grad} g$ outside Γ_2. Such a vector field is

$$Y(x,y) = 2(\mu(x,y)x, -y),$$

where $\mu : \mathbb{R}^k \times \mathbb{R}^{n-k} \to [0,1]$ is a smooth function which is 0 on Γ_1, and equals 1 outside Γ_2. Each flow line of Y beginning at a point of $(\Gamma_2 - \Gamma_1) \cap g^{-1}[-\varepsilon, \varepsilon]$ must reach $g^{-1}(-\varepsilon)$ since $|x|$ is non-decreasing along flow lines.

The global deformation of $f^{-1}[-\varepsilon,\varepsilon]$ into $f^{-1}(-\varepsilon) \cup e^k$ is obtained by moving each point of Γ at constand speed along the flow line of Y until it reaches $g^{-1}[-\varepsilon,\varepsilon] \cup B^k$ in unit time and transporting this motion to M via ϕ. Each point of $f^{-1}[-\varepsilon,\varepsilon] - \phi^{-1}(\Gamma)$ moves at constant speed along the flow line of $-\operatorname{grad} f$ until it reaches $f^{-1}(-\varepsilon)$ in unit time. Points on $f^{-1}(-\varepsilon) \cup e^k$ stay fixed. □

Definition 3.4. The k-th *type number* of a Morse function $f : M \to \mathbb{R}$ is the number $\nu_k = \nu_k(f)$ of critical points of index k, where $0 \le k \le n = \dim M$. We say that f has *type* $(\nu_0, \dots, \nu_n)$.

Theorem 3.5. *Let $f : M \to [a,b]$ be an admissible Morse function of type $(\nu_o, \dots, \nu_n)$ on a compact manifold. Suppose f has just one critical value c, $a < c < b$. Then there are disjoint k-cells $e_i^k \subset M - f^{-1}(b)$, $1 \le i \le \nu_k$, $k = 0, \dots, n$, such that $e_i^k \cap f^{-1}(a) = \partial e_i^k$, and there is a deformation retraction of M onto*

$$f^{-1}(a) \cup \bigcup_{i,k} e_i^k.$$

The proof is the same as the proof of Theorem 3.3, except that one uses disjoint Morse charts for each critical point.

4. CW Complexes

The following facts about CW complexes and attaching maps can be found in Milnor's *Morse Theory* on page 21 (although in a slightly less general form). Another good reference is *The Topology of CW Complexes* by Lundell and Weingram.

Proposition 4.1. *Let $f : X \to Y$ be a homotopy equivalence, let (A, B) be a CW pair, and let $\phi : B \to X$ be a map. Then $X \cup_\phi A$ is homotopy equivalent to $Y \cup_{f\circ\phi} A$.*

Definition 4.2. A map $f : X \to Y$ between CW complexes is *cellular* if for all $n \ge 0$, f maps the n-skeleton of X to the n-skeleton of Y, that is, $f(X^n) \subseteq Y^n$.

Proposition 4.3. *Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.*

Proposition 4.4. *Let (A, B) be a CW pair and let $\phi, \psi : B \to X$ be two maps. If ϕ are ψ are homotopic, then the adjunction spaces $X \cup_\phi A$ and $X \cup_\psi A$ are homotopy equivalent.*

Proposition 4.5. *Let X be a CW complex, let (A, B) be a CW pair, and let $f : B \to X$ be a cellular map. Then the adjunction space $X \cup_f A$ is a CW complex.*

Theorem 4.6. *Suppose X has the homotopy type of a CW complex and let (A, B) be a CW pair. Then for any map $\phi : B \to X$, $X \cup_\phi A$ has the homotopy type of a CW complex.*

Proof. By assumption there exists a CW complex Y and a homotopy equivalence $f : X \to Y$. Then by Proposition 4.1, $X \cup_\phi A$ is homotopy equivalent to $Y \cup_{f\circ\phi} A$. By Proposition 4.3, $f \circ \phi$ is homotopic to a cellular map $\psi : B \to Y$ and this induces a homotopy equivalence between $Y \cup_{f\circ\phi} A$ and $Y \cup_\psi A$ by Proposition 4.4. Finally, $Y \cup_\psi A$ is a CW complex by Proposition 4.5, proving the result. □

Theorem 4.7. *Let M be a compact n-manifold and $f : M \to [a, b]$ and admissible Morse function of type $(\nu_0, \ldots, \nu_n)$ such that $\partial M = f^{-1}(b)$. Then M has the homotopy type of a finite CW complex having exactly ν_k cells of each dimension $k = 0, \ldots, n$ and no other cells.*

Proof. The proof is by induction on the number of critical values of f. If c_1 is the smallest critical value, then c_1 is the absolute minimum of f. This follows from the fact that f attains an absolute minimum since M is compact, and if $d < c_1$ is the absolute minimum, d is a regular value of f since c_1 was assumed to be the smallest critical value. But by the local form of submersions, f maps onto an open neighborhood of d, a contradiction.

So choose $a_1 > c_1$ so that c_1 is the only critical value in $[a, a_1]$. Then $f^{-1}[a, a_1]$ has the homotopy type of a finite discrete set of points by Theorem 3.5, hence the homotopy type of a CW complex. This starts the induction. Now assume that $c_n \in (a_{n-1}, a_n)$ is the only critical value in the interval $[a_{n-1}, a_n]$, and that $f^{-1}[a, a_{n-1}]$ has the homotopy type of a CW complex. By Theorem 3.5, $f^{-1}[a_{n-1}, a_n]$ deformation retracts onto $f^{-1}(a_{n-1}) \cup \bigcup_{i,m} e_i^m$. By holding $f^{-1}[a, a_{n-1}]$ fixed, this induces a deformation retraction of $f^{-1}[a, a_n]$ onto $f^{-1}[a, a_{n-1}] \cup \bigcup_{i,m} e_i^m$, which has the homotopy type of a CW complex by Theorem 4.6. This finishes the inductive step. □