Simplicial Approximation Theorem[1]

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Definition: A subset A of euclidean space is called **affine** if, for every pair of distinct points $x, x' \in A$, the line determined by x, x' is contained in A.

Definition: An **affine combination** of points $p_0, p_1, \cdots, p_m$ in $\mathbb{R}^n$ is a point x with

$$x = t_0p_0 + t_1p_1 + \cdots + t_mp_m, \qquad \text{where } \sum_{i=0}^{m} t_i = 1$$

. **A convex combination** is an affine combination for which $t_i \geq 0$ for all i.

Definition: An ordered set of points $\{p_0, p_1, \cdots, p_m\} \subset \mathbb{R}^n$ is **affine independent** if $\{p_1 - p_0, p_2 - p_0, \cdots, p_m - p_0\}$ is a linearly independent subset of the real vector space $\mathbb{R}^n$.

Remark: Any linearly independent subset of $\mathbb{R}^n$ is an affine independent set; the converse is not true.

Definition: Let $\{p_0, p_1, \cdots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set spanned by this set, denoted by $[p_0, p_1, \cdots, p_m]$, is called the (affine) $m-$**simplex** with **vertices** $p_0, p_1, \cdots, p_m$.

Definition: If $\{p_0, \cdots, p_m\}$ is affine independent, the **barycenter** of $[p_0, \cdots, p_m]$ is $(1/m+1)(p_0 + p_1 + \cdots + p_m)$.

Definition: Let $[p_0, p_1, \cdots, p_m]$ be an $m-$simplex. The **face opposite** p_i is

$$[p_p, \cdots, \hat{p_i}, \cdots, p_m] = \left\{ \sum t_jp_j : t_j \geq 0, \sum t_j = 1, \text{ and } t_i = 0 \right\}$$

($\hat{}$ means "delete"). The **boundary** of $[p_0, p_1, \cdots, p_m]$ is the union of its faces.

Definition: Let S denote the $n-$simplex $[p_0, \cdots, p_n]$, **diam** $S = \sup_{i,j} ||p_i - p_j||$.

Definition: If $s = [v_0, v_1, \cdots, v_q]$ is a $q-$simplex, then we denote its vertex set by $\text{Vert}(s) = \{v_0, \cdots, v_q\}$.

Definition: If s is a simplex, then a **face** of s is a simplex s' with $\text{Vert}(s') \subset \text{Vert}(s)$; one writes $s' \leq s$. If $s' < s$ (i.e. $\text{Vert}(s') \subsetneq \text{Vert}(s)$), then s' is called a **proper face** of s.

Definition: A finite **Simplicial complex** K is a finite collection of simplexes in some euclidean space such that:

(i) if $s \in K$, then every face of s also belongs to K;

(ii) if $s, t \in K$, then $s \cap t$ is either empty or a common face of s and of t.

We write $\text{Vert}(K)$ to denote **vertex set** of K, namely, the set of all $0-$simplexes in K.

Definition: If K is a simplicial complex, its **underlying space** $|K|$ is the subspace (of the ambient euclidean space)

$$|K| = \bigcup_{s \in K} s,$$

the union of all simplexes in K.

Remark: Clearly, $|K|$ is a compact subspace of some euclidean space. Note that if s is a simplex in K, then $|s| = s$.

Definition: A topological space X is a **polyhedron** if there exists a simplicial complex K and a homeomorphism $h : |K| \to X$. The ordered pair (K, h) is called a **triangulation** of X.

Definition: If K is the family of all proper faces of an $n-$simplex s, then there is a triangulation (K, h) of S^{n-1}. Denote this simplicial complex K by $\dot{s}$.

Definition: Let s be a $q-$simplex. If $q = 0$, define $s^o = s$; if $q > 0$, define $s^o = s - \dot{s}$. One calls s^o an **open $q-$simplex**.

Definition: Let K be a simplicial complex and let $p \in \text{Vert}(K)$. Then the **star** of p, denoted by $\text{st}(p)$, is defined by

$$\text{st}(p) = \bigcup_{\substack{s \in K \\ p \in \text{Vert}(s)}} s^o \subset |K|.$$

Definition: If K is a simplicial complex, define its **dimension**, denoted by $\dim K$, to be

$$\dim K = \sup_{s \in K} \{\dim s\}$$

(a $q-$simplex has dimension q).

<u>Definition</u>: Let K and L be simplicial complexes. A **simplicial map** $\varphi : K \to L$ is a function $\varphi : \text{Vert}(K) \to \text{Vert}(L)$ such that whenever $\{p_0, p_1, \cdots, p_q\}$ spans a simplex of K, then $\{\varphi(p_0), \varphi(p_1), \cdots, \varphi(p_q)\}$ spans a simplex of L.

<u>Definition</u>: A map of the form $|\varphi| : |K| \to |L|$, where $\varphi : K \to L$ is a simplicial map, is called **piecewise linear**.

Let $\text{Vert}(K) = \{p_0, p_1, \cdots, p_n\}$. Every point $x \in |K|$ belongs to the interior of exactly one simplex in K. Let $s = [p_0, p_1, \cdots, p_k]$ be the simplex. We have $x = \sum_{i=1}^{k} \lambda_i p_i$ with $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i > 0$ for all i. Setting $b_i(x) = \lambda_i$ for $0 \leq i \leq k$ and $b_i(x) = 0$ for $k+1 \leq i \leq n$ we have $x = \sum_{i=0}^{n} b_i(x) p_i$.

$$|\varphi|(x) = \sum_{i=0}^{n} b_i(x)\varphi(p_i).$$

<u>Definition</u>: Let K and L be simplicial complexes, let $\varphi : K \to L$ be simplicial map, and let $f : |K| \to |L|$ be continuous. Then φ is a **simplicial approximation** to f if, for every vertex p of K,

$$f(\text{st}(p)) \subset \text{st}(\varphi(p)).$$

Remark: It is easy to see that $|\varphi|(\text{st}(p)) \subset \text{st}(\varphi(p))$. Thus we are saying that f behaves like $|\varphi|$ in that it carries neighboring simplexes of p inside the union of the simplexes near $\varphi(p)$.

<u>Definition</u>: If s is a simplex, let b^s denote its barycenter. If K is a simplicial complex, define Sd K, the **barycentric subdivision** of K, to be the simplicial complex with

$$\text{Vert}(\text{Sd } K) = \{b^s : s \in K\}$$

and with simplexes $[b^{s_0}, b^{s_1}, \cdots, b^{s_q}]$, where the s_i are simplexes in K with $s_0 < s_1 < \cdots < s_q$.

<u>Definition</u>: If K is a simplicial complex, then

$$\textbf{mesh } K = \sup_{s \in K}\{\text{diam}(s)\},$$

where $\text{diam}(s)$ denotes the diameter of s.

<u>Definition</u>: A **subcomplex** L of a simplicial complex K is a simplicial complex contained in K (i.e., $s \in L$ implies that $s \in K$) with $\text{Vert}(L) \subset \text{Vert}(K)$.

<u>Definition</u>: For any $q \geq -1$, the q−**skeleton** of K, denoted by $K^{(q)}$, is the subcomplex of K consisting of all simplexes $s \in K$ with $\dim(s) \leq q$.

Lemma 1: If $p_0, p_1, \cdots, p_n \in \text{Vert}(K)$, then $\{p_0, p_1, \cdots, p_n\}$ spans a simplex of K if and only if $\bigcap_{i=0}^{n} \text{st}(p_i) \neq \emptyset$.

Proof:

($\Rightarrow$) Assume $\{p_0, p_1, \cdots, p_n\}$ spans a simplex s of K.
Then $s^o \neq \emptyset$ and $s^o \subset \text{st}(p_i)$ for all $i = 0, \cdots, n$.
Then s^o is in the intersection.

($\Leftarrow$) Suppose $\bigcap_{i=0}^{n} \text{st}(p_i) \neq \emptyset$.
Then $\exists s^o \subset |K|$ such that $s^o \subset \text{st}(p_i)$ for all $i = 0, \cdots, n$.
Then $p_0, p_1, \cdots, p_n \in \text{Vert}(s)$
Since $p_0, p_1, \cdots, p_n \in \text{Vert}(K)$, $[p_0, p_1, \cdots, p_n]$ is a face of some simplex of K.
Therefore, $[p_0, p_1, \cdots, p_n]$ is a simplex of K.

Lemma 2: If mesh $K = \mu$ and $p \in \text{Vert}(K)$, then $\text{diam}(\text{st}(p)) \leq 2\mu$.

Proof:
For any point $x \in \text{st}(p)$, $|x - p| \leq \mu$, because x and p are in the same simplex.
Then for any $x, y \in \text{st}(p)$,

$$|x - y| \leq |x - p| + |p - y| \leq 2\mu.$$

Lemma 3: If dim $K = n$, then

(i)
$$\text{mesh Sd } K \leq ({}^{n}\!/_{n+1}) \text{ mesh } K.$$

(ii) For $q \geq 1$,
$$\text{mesh Sd}^q\ K \leq ({}^{n}\!/_{n+1})^q \text{ mesh } K.$$

Proof:

(i) Let $s = [b^{s_o}, b^{s_1}, \cdots, b^{s_q}]$, where $b^{s_i} \in \text{Vert(Sd } K)$ and s_i are simplexes in K with $s_0 < s_1 < \cdots < s_q$, $i = 1, \cdots, q$.
Then $\text{diam}(s) = \sup\limits_{i,j} ||b^{s_i} - b^{s_j}||$.
WLOG, let $i < j$, then

$$||b^{s_i} - b^{s_j}|| \leq \frac{n_j}{n_j + 1}\text{diam}(s_j), \qquad \text{(using theorem 2.9)}$$

where $n_j = \dim s_j$.
Since $s_j \in K$, $\text{diam}(s_j) \leq \text{mesh } K$

Also since $n_j \leq n$, we have $\dfrac{n_j}{n_j + 1} \leq \dfrac{n}{n+1}$.

Then $\text{diam}(s) \leq \dfrac{n}{n+1}\text{mesh } K$.

Therefore, $\text{mesh Sd } K \leq \dfrac{n}{n+1}\text{mesh } K$.

(ii) Clear for $q = 1$. By induction, assume that the statement is true for $q - 1$.
Let $s = [b^{s_0}, b^{s_1}, \cdots, b^{s_m}]$, where $b^{s_j} \in \text{Vert}\,(\text{Sd}^q\ K)$ and s_j are simplexes in $\text{Sd}^{q-1}\ K$ with $s_0 < s_1 < \cdots < s_m$, $j = 1, \cdots, m$. Using the same argument in part (i),

we get
$$\begin{aligned}\text{mesh Sd}^q\ K &\leq \frac{n}{n+1}\text{mesh Sd}^{q-1}\ K \\ &\leq \left(\frac{n}{n+1}\right)\left(\frac{n}{n+1}\right)^{q-1} \text{mesh } K \\ &= \left(\frac{n}{n+1}\right)^{q} \text{mesh } K\end{aligned}$$

Theorem: (Simplicial Approximation Theorem) *If K and L are simplicial complexes and if $f : |K| \to |L|$ is continuous, then there is an integer $q \geq 1$ and a simplicial approximation $\varphi : Sd^q\ K \to L$ to f.*

Proof:

Let $\mathrm{Vert}(L) = \{w_j : j \in J\}$ and let $\{\mathrm{st}(w_j)\}$ be the open cover of $|L|$ by its stars.
Since f is continuous, $\{f^{-1}\mathrm{st}(w_j)\}$ is an open cover of $|K|$.
Now, since $|K|$ is compact metric, this cover has a Lebesgue number $\lambda > 0$.
By Lemma 3, we can choose q large enough so that mesh $\mathrm{Sd}^q\ K < \frac{1}{2}\lambda$.
Then by Lemma 2, $\mathrm{diam}(\mathrm{st}(p)) < \lambda$ for every $p \in \mathrm{Vert}(\mathrm{Sd}^q\ K)$.

Define $\varphi : \mathrm{Vert}(\mathrm{Sd}^q\ K) \to \mathrm{Vert}(L)$ by $\varphi(p) = w_j$, where w_j is some vertex with $\mathrm{st}(p) \subset f^{-1}(\mathrm{st}(w_j))$.
Such a w_j exists because $\mathrm{diam}(\mathrm{st}(p)) < \lambda$. If there are more than one such w_j, then pick any one.

Then $f(\mathrm{st}(p)) \subset \mathrm{st}(w_j) = \mathrm{st}(\varphi(p))$.

Need to show: φ is a simplicial map. i.e. if $\{p_0, p_1, \cdots, p_m\}$ spans a simplex in $\mathrm{Sd}^q K$, then $\{\varphi(p_0), \varphi(p_1), \cdots, \varphi(p_m)\}$ spans a simplex in L.

Now if $\{p_0, p_1, \cdots, p_m\}$ spans a simplex in $\mathrm{Sd}^q\ K$, then by Lemma 1, $\bigcap_{i=0}^{m} \mathrm{st}(p_i) \neq \emptyset$. Then

$$\emptyset \neq f\left(\bigcap_{i=0}^{m} \mathrm{st}(p_i)\right) \subset \bigcap_{i=0}^{m} f\left(\mathrm{st}(p_i)\right) \subset \bigcap_{i=0}^{m} \mathrm{st}\left(\varphi(p_i)\right).$$

Then again by Lemma 1, $\{\varphi(p_0), \varphi(p_1), \cdots, \varphi(p_m)\}$ spans a simplex in L.

Proposition: (Quantitative Simplicial Approximation Theorem) *For finite simplicial complexes X and Y with piecewise linear metrics, there are constants C and C' such that any $L-$Lipschitz map $f : X \to Y$ has a $CL-$Lipschitz simplicial approximation via a homotopy of thickness $CL + C'$ and width C'.* [2]

Corollary: *Let K and L be simplicial complexes, and let $f : |K| \to |L|$ be continuous. Assume that K' is a simplicial complex such that*

(i) $|K'| = |K|$;

(ii) $\mathrm{Vert}(K) \subset \mathrm{Vert}(K')$;

(iii) mesh K' is "small".

then there exists a simplicial approximation $\varphi : K' \to L$ to f.

Proof:

Here we repeat the proof of the above theorem.
Let $\mathrm{Vert}(L) = \{w_j : j \in J\}$ and let $\{\mathrm{st}(w_j)\}$ be the open cover of $|L|$ by its stars.
Since f is continuous, $\left\{f^{-1}\mathrm{st}(w_j)\right\}$ is an open cover of $|K|$.
Now, since $|K|$ is compact metric, this cover has a Lebesgue number $\lambda > 0$.
Since mesh K' is small, we can say that mesh $K' < \frac{1}{2}\lambda$.
Then by Lemma 2, $\mathrm{diam}(\mathrm{st}(p)) < \lambda$ for every $p \in \mathrm{Vert}(K')$.

Define $\varphi : \mathrm{Vert}(K') \to \mathrm{Vert}(L)$ by $\varphi(p) = w_j$, where w_j is some vertex with $\mathrm{st}(p) \subset f^{-1}(\mathrm{st}(w_j))$.

Then $f(\mathrm{st}(p)) \subset \mathrm{st}(w_j) = \mathrm{st}(\varphi(p))$.

To show that φ is a simplicial map, let $\{p_0, p_1, \cdots, p_m\}$ spans a simplex in K'.
By Lemma 1, $\bigcap_{i=0}^{m} \mathrm{st}(p_i) \neq \emptyset$. Then

$$\emptyset \neq f\left(\bigcap_{i=0}^{m} \mathrm{st}(p_i)\right) \subset \bigcap_{i=0}^{m} f\left(\mathrm{st}(p_i)\right) \subset \bigcap_{i=0}^{m} \mathrm{st}\left(\varphi(p_i)\right).$$

Then again by Lemma 1, $\{\varphi(p_0), \varphi(p_1), \cdots, \varphi(p_m)\}$ spans a simplex in L.

Lemma 4: *If $\varphi : K \to L$ is a simplicial map, then $\varphi\left(K^{(q)}\right) \subset L^{(q)}$ for every q. Therefore,* $\dim K = n$ *implies that* $\mathrm{im}|\varphi| \subset |L^{(n)}|$.

Proof:

If $s \in K^{(q)}$, then $s = [p_0, \cdots, p_r]$ for some $r \leq q$.
Then $\varphi(s) = [\varphi(p_0), \cdots, \varphi(p_r)] \in L^{(q)}$.
So, $\varphi(K^{(q)}) \subset L^{(q)}$.

Then $\varphi(K) \subset L^{(n)}$ when $\dim K = n$.
For $x \in |K|$, $x = \sum_{i=0}^{n} \lambda_i p_i$ and $|\varphi|(x) = \sum_{i=0}^{n} b_i(x)\varphi(p_i)$ where $p_i \in \mathrm{Vert}(K)$.
Then $|\varphi|(x) \in |L^{(n)}|$. So, $\mathrm{im}|\varphi| \subset |L^{(n)}|$.

Lemma 5: *If $\varphi : K \to L$ is a simplicial approximation to $f : |K| \to |L|$, then $|\varphi| \simeq f$.*

Proof:

Enough to show that $\forall x \in |K|$, $|\varphi|(x)$ and $f(x)$ are in the same simplex. Then the map $H(t, x) = (1-t)|\varphi|(x) + tf(x)$ where $t \in [0, 1]$ is the desired homotopy.

Need to show: whenever $x \in |K|$ and $f(x) \in s^o$ where s is a simplex of L, then $|\varphi|(x) \in s$.

Let $x \in t^o$, where $t = [p_0, p_1, \cdots, p_q]$ is a simplex in K.
Then $x \in \mathrm{st}(p_i)$ for some $i = 1, \cdots, q$.
Which implies $f(x) \in \mathrm{st}\left(\varphi(p_i)\right)$.
So, $s^o \subseteq \mathrm{st}\left(\varphi(p_i)\right)$.
Then $\varphi(p_i)$ is a vertex of s for each $i = 1, \cdots, q$.
Since $|\varphi|(x)$ is determined by $\varphi(p_i)$, $|\varphi|(x) \in s$.

Theorem: *If $m < n$, then every continuous map $f : S^m \to S^n$ is nullhomotopic.*

Proof:

Let K be the $m-$skeleton of an $(m+1)$ simplex, and let L be the $n-$skeleton of an $(n+1)$ simplex.
We may regard f as a continuous map from $|K|$ into $|L|$.
By the Simplicial Approximation Theorem, let $\varphi : \mathrm{Sd}^q\ K \to L$ be a simplicial approximation to f.
Since $\dim \mathrm{Sd}^q\ = \dim K = m$, by Lemma 4, $\mathrm{im}|\varphi| \subset |L^{(m)}|$.
Hence $|\varphi|$ is not surjective. In particular $\mathrm{im}|\varphi| \subset |L| - \{\text{point}\}$, which is contractible.
So, $|\varphi|$ is nullhomotopic. But by Lemma 5, $|\varphi| \simeq f$.
Therefore, f is nullhomotopic.

References

[1] Joseph J. Rotman. *An Introduction to Algebraic Topology*. Springer, GTM 119, 1998.

[2] Gregory R Chambers. Quantitative nullhomotopy and rational homotopy type. *Geometric and Functional Analysis*, 28(3), 2018.