HOMOLOGY OF PSEUDOMANIFOLDS

Definition. A finite simplicial complex M is an n-dimensional *pseudomanifold* if:

(a) Any simplex of M is a face of an n-dimensional simplex;

(b) Any $(n-1)$-simplex of M is a common face of *exactly two* simplices (no 'branching').

(c) Any two n-simplices s, t of M can be connected by a chain of n-simplices $s_0 = s, s_1, \dots s_k = t$ ($s_i \cap s_{i+1}$ is a common $(n-1)$-dimensional face.

Remark 1. In particular a point in an open n-simplex or $(n-1)$ simplex has a nbd. homeomorphic to R^n ('manifold point'). The complement of this set in M is a closed set of (Hausdorff) dimension $\leq n-2$.

Remark 2. Any triangulable manifold (in particular, any topological manifold of dimension ≤ 3, or differentiable manifolds of any dimension) admits a triangulation with a pseudomanifold structure. To see that not every pseudomanifold is a manifold, consider a triangulated orientable surface, and the quotient space obtained by identifying two vertices that are not vertices of the same 2-simplex. The quotient inherits a natural triangulation, which can be seen to have a pseudomanifold structure.

Remark 3. Pseudomanifolds admit a similar definition in the setting of CW complexes. Recall a CW complex is *regular* if for each n-cell e^n, $n > 0$, there exists a characteristic map $\Phi : D^n \to \overline{e^n}$ which is a homeomorphism (D^n is the closed n-disk.) That is, we require the attaching map to be a homeomorphism of S^{n-1} into the $(n-1)$-skeleton. A finite, regular, n-dimensional CW complex is an n-dimensional pseudomanifold if it satisfies: (1) every cell is a face of some n-cell; (2) every $(n-1)$-cell is the face of exactly two n-cells; (3) Given any two n-cells e^n_a, e^n_b, there exist a chain of n-cells $e^n_0 = e^n_a, \dots e^n_k = e^n_b$, so that e^n_{i-1} and e^n_i have a common $(n-1)$-dimensional face (possibly more than one.)

It can be shown that this condition on a CW space is topological, that is, independent of a particular cell decomposition. (For more details on the CW complex approach, see [Massey, p.249]).

Remark 4. (See [Fuks-Viro-Rokhlin p. 159.]) Not every homology class of a smooth manifold can be realized by a smooth submanifold. This is the classical *Steenrod problem*, on which a breakthrough was made by [R. Thom, 1954]: for any homology class, an odd multiple can be realized by a submanifold; any homology class in codimension two or greater can be realized by a smooth submanifold; and every homology class may be realized by a submanifold with singularities of codimension ≥ 2, in particular by pseudomanifolds (when triangulated.) Hence the interest of this topic.

Notation: in what follows, $\mathbf{s}$ denotes a simplex s with a given orientation.

Theorem 1. Let M be an n-dimensional pseudomanifold. With $\mathbb{Z}_2$ coefficients: $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$.

With $\mathbb{Z}$ coefficients: $H_n(M) = \mathbb{Z}$ (orientable case) or $H_n(M) = 0$ (unorientble.)

Proof. With $\mathbb{Z}_2$ coefficients. Let $\Gamma = \sum_{s \in M} s \in C_n(M, \mathbb{Z}_2)$, the sum of all n-dimensional simplices in M. Due to (b), $\partial\Gamma = 2\sum_{t \in M} t$ (sum over all $(n-1)$-simplices of M), so $\partial\Gamma = 0$ (mod 2). Thus $\Gamma \in Z_n(M; \mathbb{Z}_2)$ (not homologous to 0).

Now, any $x \in C_n(M; \mathbb{Z}_2)$ has the form $x = s_1 + \ldots + s_k$, where the s_i are n-simplices. If $\partial x = 0$ and s is a term in this sum, ay adjacent n-simplex s' must also occur (to cancel in ∂x the coefficients of their common $(n-1)$-face.) Thus (from condition (c)) either $x = 0$ or $x = \Gamma$. Thus $H_n(M; \mathbb{Z}_2) = Z_n(M; \mathbb{Z}_2) = \mathbb{Z}_2$, with Γ as a generator.

With $\mathbb{Z}$ coefficients. Suppose we may choose an orientation $\mathbf{s}$ for each n-simplex of M so that adjacent n-simplices induce opposite orientations on the common face ('M is orientable'.) Then the chain $\Gamma = \sum_{s \in M} \mathbf{s}$ is a *cycle* (not homologous to 0). And if $z = \sum_{s \in M} m_s \mathbf{s}$ is a cycle, we must have $m_s = m_{s'}$ if s, s' are adjacent (since $\partial z = \sum_t k_t \mathbf{t} = \mathbf{0}$ implies $k_t = 0 \forall t$, by (b)). Again it follows that $m_s = m_{s'}$ for all n-simplices s, s'; so $z = m\Gamma$ for some $m \in \mathbb{Z}$. Thus $H_n(M) = \mathbb{Z}$.

If M is not orientable: Let $z = \sum_{s \in M} m_s \mathbf{s} \in Z_n(M; \mathbb{Z})$. Then $\partial z = \sum_{t \in M} k_t \mathbf{t} = \mathbf{0}$ (sum over all $(n-1)$ simplices), so $k_t = 0 \forall t$. Thus if $t = s \cap s'$: $0 = k_t = \pm m_s \pm m_{s'}$ (signs depending on the orientations induced by $\mathbf{s}, \mathbf{s}'$ on t.) So $|m_s| = |m_{s'}|$ for any adjacent s, s', and hence for any $s, s' \in M$.

Thus z is a sum of terms like $m\mathbf{s}$ and $-m\mathbf{s}'$ (same m for all s), so changing the orientation of some s, we have $z = m \sum_{s \in M} \mathbf{s}$. Since $\partial z = 0$, $m = 0$ or $\sum_{s \in M} \mathbf{s}$ is an n-cycle, so with the new orientation any $(n-1)$ simplex inherits opposite orientations from the n-simplices incident to it (and M would be orientable). Thus $m = 0$, i.e. $H_n(M; \mathbb{Z}) = 0$.

Theorem 2. Let M be an n-dimensional pseudomanifold.

If M is orientable, $H_{n-1}(M; \mathbb{Z})$ is free (no torsion.)

If M is non-orientable, H_{n-1} has a unique torsion element $\alpha \neq 0$, with $2\alpha = 0$; i.e. $H_{n-1}(M;\mathbb{Z}) = F \oplus \mathbb{Z}_2$, F free.

Proof. (1) Assume M is orientable. By contradiction, let $y \in C_{n-1}(M)$ is such that for some $p \geq 0$ we have $py = \partial x$. Fix an orientation for each n-simplex s, such that $\Gamma = \sum_{s\in M} s$ generates $H_n(M)$. Pick also (arbitrarily) an orientation for each $(n-1)$-simplex t. So:

$$x = \sum_s m_s \mathbf{s}, \qquad y = \sum_t k_t \mathbf{t}, \qquad (m_s, k_t \in \mathbb{Z}).$$

Any adjacent n-simplices $\mathbf{s}', \mathbf{s}''$ induce opposite orientations on the common $(n-1)$-face $t = s' \cap s''$. From $\partial x = py$ follows $m_{s'} - m_{s''} = \pm pk_t$, or $m_s \equiv m_{s'} \pmod p$ if s', s'' are adjacent. From condition (c) it follows that all coefficients m_s are pairwise congruent mod p; i.e. there exists $0 \leq r < p$ such that for all s: $m_s = pq_s + r$. Then:

$$x = \sum_s m_s \mathbf{s} = p \sum_s q_s \mathbf{s} + r \sum_s \mathbf{s} = px' + r\Gamma,$$

where $x' = \sum_s q_s \mathbf{s}$. So $py = \partial x = p\partial x'$ (since $\partial\Gamma = 0$), or $y = \partial x'$.

We conclude that $py = \partial x$ implies $[y] = 0$, so $H_{n-1}(M)$ is torsion-free.

(2) Assume M is nonorientable. Fix arbitrary orientations on the n-simplices, and let $x = \sum_s \mathbf{s}$. Then $\partial X = 2a$, where $a = \sum_t \mathbf{t}$ is the sum of the $(n-1)$-dimensional oriented simplices $\mathbf{t}$ which inherit the same orientation from 2 adjacent n-simplices meeting at t. Note $a \neq 0$ (since M is non-orientable) and $a \in Z_{n-1}(M)$, since $2\partial a = 2\partial\partial x = 0$, but a is not a boundary: if $a = \partial x'$, then $\partial x = 2\partial x'$, or $\partial(x - 2x') = 0$. So $x = 2x'$ (since $H_n(M) = 0$); impossible, since the coefficients of x in the basis $\{s; s \text{ an n-simplex of } M\}$ of $C_n(M)$ are all 1 (and we use $\mathbb{Z}$ coefficients.)

We conclude $[a] \neq 0$ in $H_{n-1}(M,\mathbb{Z})$ (And $2[a] = 0$.)

Remark. Note $[a]$ is independent of the choice of orientations of the n-simplices s. If instead we consider $x' = \sum_s \mathbf{s}'$ (picking different orientations for the s) , we have $x - x' = 2\sum \mathbf{s}$ (over some set of simplices), so $2(a-a') = \partial(x - x') = 2\partial \sum \mathbf{s}$, so $[a] = [a']$.

Claim. $[a]$ is the only element of finite order in $H_{n-1}(M;\mathbb{Z})$.

Proof. Let $y \in Z_{n-1}(M,\mathbb{Z})$ such that $py = \partial x$, for some $0 \neq p \in \mathbb{Z}_+$. So:

$$x = \sum_s m_s \mathbf{s}, \quad y = \sum_t k_t \mathbf{t}, \quad py = \sum_t pk_t \mathbf{t}.$$

For each $(n-1)$-simplex $t = s \cap s'$, $py = \partial x$ implies $pk_t = m_s \pm m_{s'}$, so $m_{s'} \equiv \pm m_s \pmod{p}$ for any adjacent n-simplices, and hence (by condition (c)) for any two n-simplices. Changing n-simplex orientations if needed, we may assume $m_s \equiv m_{s'} (p)$ for all s, s'. As before, write: $m_s = pq_s + r, 0 \leq r < p$ (r indep. of s), so:

$$x = \sum_s m_s \mathbf{s} = p \sum_s q_s \mathbf{s} + r \sum_s \mathbf{s} = px' + r\gamma, \quad \gamma := \sum_{s \in M} \mathbf{s},$$

the sum over all n-simplices. As before, we have $\partial\gamma = 2a$, where $0 \neq a = \sum \mathbf{t}$, the sum over the $(n-1)$-simplices t on which the 2 adjacent n-simplices induce opposite orientations; recall $2[a] = 0$.

We *claim* $[y] = q[a]$ for some q, and hence $q = 0$ or $q = 1$ (since $[a]$ has order 2). So either $[y] = 0$ or $[y] = [a]$.

Proof. From $\partial\gamma = 2a$ follows:

$$py = \partial x = p\partial x' + r\partial\gamma = p\partial x' + 2ra,$$

or $p(y - \partial x') = 2ra$. Since a is a sum of simplices with coefficients 1, it follows that $2r$ is a multiple of p. So:

$$p(y - \partial x') = pqa, \text{ or } y - \partial x' = qa, \text{ or } [y] = q[a].$$