Problems from Hatcher

srengasw

September 2021

1 Problem 16: S^∞ is contractible

This is remarkable, because none of the S^n's are contractible for any $n \geq 0$. (One way to see this is that S^0 has two connected components, S^1 has a nontrivial fundamental group, and spheres of higher dimension have nontrivial homology groups $H^n(S^n)$. On the other hand, contractible spaces have the homology and homotopy of a point.) However, the reason S^∞ is contractible is because of the simple fact that S^n can be contracted to a point in S^{n+1}, e.g. the equator S^1 of S^2 can be shrunk to the north pole.

The intuitive idea here is to "move" all of S^∞ into its "equator", and then shrink it to a point. First, we note that S^∞ contains a (homeomorphic) copy of itself as a nontrivial subspace. To see this, recall that one of the ways to realise S^∞ is through a sequence of inclusions $S^0 \xhookrightarrow{(id,0)} S^1 \xhookrightarrow{(id,0)} S^2 \hookrightarrow \ldots$. By this we mean S^n is embedded in S^{n+1} as $\{x \in S^{n+1} : x_{n+2} = 0\} \subset \mathbb{R}^{n+2}$. That is, we view S^0 as the set of poles of S^1, S^1 as the "prime meridian" of S^2 and so on (as we are thinking of x_1 as the "upward" direction). But one could also realise S^n in S^{n+1} as $\{x \in S^{n+1} : x_1 = 0\} \subset \mathbb{R}^{n+2}$. Thus there is a copy of S^0 on the equator of S^1, a copy of S^1 on the equator of S^2 and so on. This allows us to view the "equator" of S^∞ as a copy of S^∞ itself.

Now we obtain a homotopy between these two configurations of S^n inside $S^{n+1} \subset \mathbb{R}^{n+2}$. Let $F^n : S^n \times [0,1] \to \mathbb{R}^{n+2}$ be the straight-line homotopy $F^n_t(x_1, ..., x_{n+1}) = (1-t)(x_1, ..., x_{n+1}, 0) + t(0, x_1, ..., x_{n+1})$. Note that $(x_1, ..., x_n)$ satisfies $\Sigma x_i^2 = 1$. It is now a simple exercise to see that $F^n_t(x) \neq 0$ for any $x \in S^n$. So we normalise to get a vector $\tilde{F}^{n-1}_t(x) \doteqdot \frac{F^n_t(x)}{|F^n_t(x)|} \in S^{n+1}$, and $\tilde{F}^n_t$ is the desired homotopy. Identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$ lets us identify the homotopy $F^{n+1}|_{S^n \times 0}$ with F^n. Thus the various F^n's are "compatible" with each other and let us define a homotopy $F_t : S^\infty \to S^\infty$ that restricts to F_n on S^n, and carries S^∞ into its equator. Note that F_0 is the identity on S^∞.

Now for each n, we can contract the equatorial S^n of S^{n+1} to its "north pole" $(1, 0, ..., 0) \in \mathbb{R}^{n+2}$ via "latitudes". The equation for this is given by $G^n : S^n \times [0,1] \to S^{n+1}$, $G^n_t(0, x) = (t, \sqrt{1-t^2}x)$, where $x \in S^n$. These G^n's are compatible with each other in the same way as the F^n's, and allow us to define the homotopy $G_t : S^\infty \to S^\infty$ that restricts to G^n on S^n

Finally we can concatenate the two homotopies as follows:

$$H_t \doteqdot \begin{cases} F_{2t} & 0 \leq t \leq 1/2 \\ G_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$$

This homotopy has the property that H_0 is the identity and H_1 maps S^∞ to the north pole, i.e. S^∞ is contractible.

2 Problem 29: The explicit formula for the homotopy extension

Let A be a CW complex and $\phi : \partial D^n \to A$ be an attaching map that attaches an n-cell to A. Let $X \doteqdot A \sqcup_\phi D^n$ be the resulting CW complex. We suppose that there is an initial map $g_0 : X \to Y$, and a homotopy $f : A \times I \to Y$ with $g_0|_A = f_0$. We need to extend f_t to all of X. In order to do so, we must define $h_t : D^n \times I \to Y$ that is "compatible" with g_0 and f in the following sense: $h_0|_{\mathrm{int} D^n} = g_0|_{\mathrm{int} D^n}$ and $h_t|_{\partial D^n}(x) = f_t|_{\phi(\partial D^n)}(\phi(x))$.

Let $P : D^n \times I \to (D^n \times 0) \cup (\partial D^n \times I)$ be the projection map that sends (x, t) to a point $P(x, t) \doteqdot (x^*, t^*)$ on $(D^n \times 0) \cup (\partial D^n \times I)$ that lies on the straight line joining $(0, 2), (x, t) \in \mathbb{R}^n \times \mathbb{R}$. Let $B = P^{-1}(\mathrm{int} D^n \times 0)$ and $C = P^{-1}(\partial D^n \times I)$, i.e. B is the subset of $D^n \times I$ that gets projected onto the base and C is the subset that gets projected to the sides. More precisely, $B = \{(x, t) \in D^n \times I : t < 2 - 2|x|\}$ and $C = \{(x, t) \in D^n \times I : t \geq 2 - 2|x|\}$. The explicit formula for P can be worked out using plane geometry to be:

$$(x^*, t^*) = \begin{cases} \left(\frac{x}{|x|}, \frac{2|x|-2+t}{|x|}\right), & (x, t) \in C \\ \left(\frac{2}{2-t} x, 0\right), & (x, t) \in B \end{cases}$$

The map h that we want is obtained piecewise as

$$h(x, t) = \begin{cases} f(\phi(x^*), t^*), & (x, t) \in C \\ g_0(\phi(x^*)), & (x, t) \in B \end{cases}$$